Speculation, Risk Premia and Expectations in the Yield Curve

ABSTRACT

An affine asset pricing model in which traders have rational but heterogeneous expectations about future asset prices is developed. We use the framework to analyze the term structure of interest rates and to perform a novel three-way decomposition of bond yields into (i) average expectations about short rates (ii) common risk premia and (iii) a speculative component due to heterogeneous expectations about the resale value of a bond. The speculative term is orthogonal to public information in real time and therefore statistically distinct from common risk premia. Empirically we find that the speculative component is quantitatively important accounting for up to a percentage point of yields, even in the low yield environment of the last decade. Furthermore, allowing for a speculative component in bond yields results in estimates of historical risk premia that are more volatile than suggested by standard Affine Gaussian term structure models which our framework nests.

A large part of the existing empirical literature analyzing the term structure of interest rates either implicitly or explicitly decomposes bond yields into risk premia and expectations about future risk-free interest rates.[1] However, both casual observation and survey evidence suggest that there is a lot of disagreement about future interest rates. In this paper we ask how this fact should change our view about what components make up bond yields. Below, we present a flexible affine asset pricing framework in which traders have rational but heterogeneous expectations about future bond yields. We use the framework to argue that in addition to the classic term structure components, heterogeneous expectations about the resale value of a bond give rise to an empirically relevant and statistically distinct third bond yield component due to speculative behavior.

In markets where assets are traded among agents who may not want to hold on to the asset until it is liquidated, expectations about the resale value of an asset will matter for its current price. When rational traders have access to different information about future fundamentals, the price of the asset deviates systematically from the "consensus value" defined as the hypothetical price that would reflect the average opinion of the (appropriately discounted) fundamental value of the asset (e.g. Allen, Morris and Shin 2006, Bacchetta and van Wincoop 2006, Nimark (2012)). These deviations from the consensus price occur because an individual trader's expectation about the resale value of an asset can with heterogeneous expectations be different from what the individual trader would be willing to pay for the asset if he were to hold it until maturity. Heterogeneous expectations then give rise to speculative behavior in the sense of Harrison and Kreps (1978), who defined such behavior to be present when the possibility of reselling an asset changes its equilibrium price.

In this paper we derive a flexible affine pricing framework for empirically quantifying the importance of the type of speculative behavior described above for the term structure of interest rates. The framework differs from most of the previous theoretical literature on asset pricing with heterogeneously informed agents in that we do not specify utility functions, nor do we model the portfolio decisions of traders explicitly.[2] Instead, we make an effort to stay as close as possible to the large empirical literature that uses affine models to study asset prices. In the standard

[1] Some examples are Joslin, Singleton and Zhu (2011), Duffee(2002), Cochrane and Piazzesi (2008) Bauer, Rudebusch and Wu (2012), Joslin, Priebsch and Singleton(2012)

[2] Some early examples of papers studying the theoretical implications of heterogeneous information on asset prices in a rational setting are Hellwig (1980), Grossman and Stiglitz (1980), Admati (1985), Singleton (1987). More recent examples include Allen, Morris and Shin (2006), Bacchetta and van Wincoop (2006) and Nimark (2012).

full information affine no-arbitrage framework, variation across time in expected excess returns is explained by variation across time in either the amount of risk or in the required compensation for a given amount of risk. Gaussian models such as the $A_0(N)$ models of Dai and Singleton (2000) or the model in Joslin, Singleton and Zhu (2011) focus on the latter and identify the price of risk as an affine function of a small number of factors that also determine the dynamics of the risk-free short rate. Similar to this approach, we specify a model in which variation in expected excess returns across individual traders, in the absence of arbitrage, must be accompanied by variation across traders in the required compensation for risk. The framework is flexible and it nests a standard affine Gaussian term structure model if the signals observed by traders are perfectly informative about the state. This facilitates comparison of our results to the large existing literature on affine term structure models. However, the framework is general and can be applied to price other classes of assets as well.

When implementing the framework empirically, we treat the individual responses in the Survey of Professional Forecasters as being representative of the bond yield expectations of traders randomly drawn from the population of the model. There is substantial dispersion of survey responses and the average cross-sectional standard deviation of the one-year-ahead forecasts of the Federal Funds Rate is approximately 40 basis points. In reality, bond prices depend on many things, including monetary policy, current attitudes towards risk and political, macroeconomic and financial market developments more generally. There is thus a vast amount of information available that could help predict future bond prices. In the set up presented here, different traders observe different signals with idiosyncratic noise components about a vector of common latent factors. The traders use these signals to form rational, i.e. model consistent, expectations about future risk-free rates and risk premia. This set up is a simple way to capture the fact that, in practice, it is too costly for traders to pay attention to all available information that could potentially help predict bond prices. With slightly different vantage points and historical experiences, traders instead tend to observe different subsets of all available information. Since the signals are about a common vector of latent factors, information sets will be highly correlated across traders, but not perfectly so. Formally, the set up is similar to the information structure in Singleton (1987), Allen, Morris and Shin (2006) and Bacchetta and van Wincoop (2006) though the vector of unobservable fundamentals is of a higher dimension here.

The traders in the model form rational, or model-consistent, expectations about future bond yields conditional on their information sets. We think that one appealing feature of a fully rational framework is that agents use the information contained in bond prices efficiently. This contrasts with some alternative approaches to introducing expectations heterogeneity in term structure models. For instance, Xiong and Yan (2010) present a model in which two groups of traders misinterpret an uninformative signal as being helpful in predicting future inflation. Since the two groups of agents have different subjective beliefs about the correlation between the uninformative signal and the inflation process, they have different subjective beliefs about future inflation and the real return on bonds. The model can generate speculative dynamics that to an econometrician appear to be time varying risk premia and the "excess" volatility of long bond yields documented by Gurkaynack, Sack, Swanson (2005). However, in Xiong and Yan's model, an econometrician conditioning only on publicly available bond prices could predict excess returns, though the traders in the model are assumed to disregard this information. This implies that we as econometricians could make larger trading profits than the traders inside the model simply by conditioning on publicly available prices.

That traders form model-consistent expectations also introduces additional restrictions on the dynamics of bond prices that are absent in models where differences in expectations are driven by exogenously specified beliefs processes and where agents "agree to disagree". Examples of this approach include modeling agents as having different degrees of overconfidence in the precision of a commonly observed signal as in Scheinkman and Xiong (2003). Chen, Joslin and Tran (2010, 2012) develop a flexible affine asset pricing framework for modeling differences in beliefs in which at least one group of traders do not form model consistent expectations. When agents "agree to disagree", the beliefs held by different groups of agents are common knowledge. Since prices reflect beliefs about the future and these beliefs are known, there is by assumption no information in prices that agents in this class of models find useful.

In the model presented below, traders extract information about both the latent factors and the expectations of other traders from endogenous bond prices. Imposing model-consistent expectations restricts the joint dynamics of bond prices and the cross-sectional dispersion of bond price expectations. In equilibrium, bond prices cannot reveal too much information since too informative bond prices would imply a counter-factually degenerate cross-sectional distribution of expectations.

Put differently, the fact that agents learn from endogenous prices imposes more restrictions than models in which bond prices by assumption are not allowed to convey any information useful to individual traders.

Another difference-in-beliefs based alternative to model expectation heterogeneity is to let traders learn rationally from prices but starting from heterogeneous priors as in Buraschi and Jiltsov (2006). However, rational learning from common signals implies that the beliefs of different traders will converge over time. This approach is thus not suitable for modeling and estimating phenomena that do not subside over time. Based on these considerations, we think modeling heterogeneous expectations as arising from individual traders observing different signals about stochastic latent variables is the most suitable approach for empirical work.

The model is populated by a continuum of traders who have dispersed expectations about future bond yields. This feature of the model makes it possible to, in addition to data on bond yields, use individual survey responses of interest rate forecasts in combination with likelihood based methods to estimate the parameters of the model. Our use of the full cross-section of survey forecast data to estimate a term structure model is in contrast to most of the existing term structure literature who typically use the median forecast.[3] The information in the cross-section of yield expectations clearly disciplines the degree of cross-sectional heterogeneity of expectations among the traders in the estimated model. Since the model implies strong joint predictions about the cross-sectional dispersion of expectations and bond price dynamics, survey data also sharpens our inference about the role that private heterogeneous information plays in determining bond price dynamics.

The main empirical contribution of the paper is to perform a novel three-way decomposition of bond yields. We show that in addition to the classic components due to risk premia and expectations about risk-free short rates, heterogeneous information introduces a third speculative term arising from individual traders rationally perceiving that average expectations about the resale value of a bond is different from their own best estimate. This difference between traders' own and the perceived average expectation of the resale value of the bond can equivalently be thought of as a higher order expectation error, i.e. an expectation about the error in other traders' forecasts. In a fully rational setting, all traders use the information available to them efficiently and it is

[3] See D'Amico, Kim and Wei (2008), Chun (2010) and Piazzesi and Schneider (2011) for examples of studies who have used survey data to estimate term structure dynamics

not possible for individual traders to predict the error in the average expectation based solely on publicly available information. The speculative component must therefore be orthogonal to all public information available in real time, such as bond prices. The speculative component in the term structure is thus not simply risk premia re-labeled: it is statistically distinct from classical risk premia which can be predicted conditional on bond yields in real time.

We use the estimated model to quantify the importance of the speculative term and we find that it accounts for a substantial part of bond yields. The speculative term is more important for medium to long-maturity yields than for short maturities. For instance, the speculative term in the 5 and 10 year bond yields accounts for up to approximately a percentage point of observed yields, even in the low nominal yield environment of the last decade. The speculative term is also present in shorter maturity bond yields, but our estimates suggest that in the sample it never accounts for more than half a percentage point of the 1 year yield.

The speculative component introduced by heterogeneous expectations is interesting in its own right, but allowing for it in bond yields also changes our estimates about the common component of risk premia, as compared to a full information model. Comparing our estimates of risk premia with those extracted using the nested model of Joslin, Singleton and Zhu (2011) we find that the common risk premia are more volatile in the model with heterogeneous information. The two time series of risk premia are positively correlated, implying that allowing for heterogeneous information mainly changes the magnitudes of time varying risk premia, rather than its qualitative properties.

The rest of paper is structured as follows. The next section presents an affine framework for modeling the term structure when traders observe different information relevant for predicting future bond prices. Section II shows how the framework can be used to decompose the term structure into the standard components, i.e. risk premia and expectations about future short rates, as well as speculative components driven by traders exploiting what they perceive to be inaccurate average expectations about future bond prices. Section III describes in more detail the empirical specification and how the model can be estimated. Section IV presents the empirical results. Section V concludes and the Appendix contains derivations and details left out of the main text.

I. An affine term structure model with heterogeneous information

In this section we describe how bond prices can be determined in an arbitrage-free framework when traders have heterogeneous information relevant for predicting future bond returns. The basic set up follows a large part of the affine term structure literature (see Duffie and Kan (1996) and Dai and Singleton (2000)) and posits that the short rate r_t is an affine function of a vector of state variables. In the standard approach, a model is completed by specifying how traders price risk by assuming a functional form for the stochastic discount factor (SDF). This SDF can then be used to price contingent claims. We will take a similar approach here, except that the SDFs must now be trader-specific to accommodate for the fact that information sets are trader-specific.

In the standard common information model, the price P_t^n of a bond with n periods to maturity is given by

$$P_t^n = E\left[M_{t+1} P_{t+1}^{n-1} \mid \Omega_t\right] \tag{1}$$

where P_{t+1}^{n-1} is the price of a $n-1$ periods to maturity bond in period $t+1$, Ω_t is the common information set in period t and M_{t+1} is the stochastic discount factor. In the absence of arbitrage, this relationship has to hold for all maturities n. In a model with heterogeneous information a similar relation holds, expect that the SDF is now trader-specific so that for all traders $j \in (0,1)$ and all maturities n the relationship

$$P_t^n = E\left[M_{t+1}^j P_{t+1}^{n-1} \mid \Omega_t^j\right] \tag{2}$$

must hold. All traders pay the same price for bonds so the left hand side of (2) is common to all traders. However, trader-specific information sets introduce heterogeneity in expectations of P_{t+1}^{n-1}. For (2) to continue to hold when expectations about P_{t+1}^{n-1} differ across traders, M_{t+1}^j must also be trader-specific.

Allowing for heterogeneously informed traders introduces a few complications in terms of specifying and solving the model, relative to the standard common information set up. The first is related to the need for agents to "forecast the forecasts of others", or what Allen, Morris and Shin (2006) has labeled the "beauty contest" aspect of asset markets with heterogeneously informed traders. When long maturity bonds are traded in secondary markets, traders need to predict what *other* traders will be willing to pay for a bond in the future. That is, when traders are price takers, the expectation of P_{t+1}^{n-1} in the equilibrium condition (2) will depend on trader j's expectation about how much other traders will be willing to pay for an $n-1$ periods to maturity bond in period $t+1$. With heterogeneous information, this may be different from what an individual trader would be willing to pay if he were to hold on to the bond until it matures.

In practical terms, the beauty contest aspect introduced by heterogeneous information implies that the factors determining the short rate are no longer a complete description of the state of the world. Instead, higher order expectations of the factors, i.e. expectations about other traders' expectations about the factors, enter as endogenous state variables. The law of motion for the higher order expectations of the factors then has to be determined jointly with bond prices since traders use the information contained in bond prices to form expectations about the unobservable factors. Heterogeneous information thus introduces an additional step in deriving a process for bond prices that is not present in the full information set up with only exogenous state variables.

The second difference relative to the common information set up is that we need to specify a functional form for the individual traders' SDFs that allows for heterogeneity in expected returns. Below we propose a form that is analogous to the standard formulation under common information and indeed, nests the standard formulation when signals reveal the exogenous state perfectly.

Solving the model implies finding a fixed point on the mapping from the process of traders' expectations to bond prices and from bond prices to the process of traders' expectations. The approach we will take is the following. First, we will conjecture a functional form for the law of motion for higher order expectations of the exogenous factors. The exogenous factors as well as the average higher order expectations of these factors are the state of the model and bond prices are conjectured to be an affine function of this extended state. These conjectures are verified to hold in equilibrium.

For a given law of motion of the state and a functional form for traders' stochastic discount factors we can derive how bond prices depend on the state just as in the standard framework. Given bond prices as a function of the state, we can solve the traders' filtering problem which in turn determines the law of motion for the (endogenous) state. The model is solved by iterating between these two steps. This section presents the main steps involved, but many of the details of how to solve the model are relegated to the Appendix.

A. The conjectured processes for bond prices and the state

Following the affine term structure literature, the one period risk-free rate r_t is an affine function of a vector of state variables x_t

$$r_t = \delta_0 + \delta_x' x_t \tag{3}$$

and x_t follows a first order vector auto regression

$$x_{t+1} = \mu^P + F^P x_t + C\varepsilon_{t+1}. \tag{4}$$

In a full information setting, we would normally proceed by specifying a functional form for the stochastic discount factor that would allow us to derive the price of a bond of any maturity as an affine function of the factors x_t. In our heterogeneous information set up the factors determining the short rate are not directly observable by the traders. Instead, traders receive signals about x_t with an idiosyncratic noise component. If long bonds are traded frequently and individual traders are price takers, individual traders' expectations about future bond prices will depend on their expectations about other traders' expectations about risk-free rates and risk premia. These higher order expectations of future risk-free rates and risk premia can be reduced to higher order expectations about the current latent factors x_t. The relevant state of the model can then be shown to be the hierarchy of higher order expectations X_t defined as

$$X_t \equiv \begin{bmatrix} x_t \\ x_t^{(1)} \\ \vdots \\ x_t^{(\bar{k})} \end{bmatrix} \tag{5}$$

with the average k order expectations $x_t^{(k)}$ defined recursively as

$$x_t^{(k)} \equiv \int E\left[x_t^{(k-1)} \mid \Omega_t^j\right] \, dj$$

The information set of trader j is denoted Ω_t^j and $\bar{k}$ is the maximum order of expectations considered. Nimark (2011) demonstrates that a finite $\bar{k}$ is sufficient to approximate the true equilibrium dynamics to an arbitrary accuracy.

We will conjecture (and later verify) that the state X_t follows a first order vector auto regression process

$$X_{t+1} = \mu^X + \mathbf{M}X_t + \mathbf{N}\mathbf{u}_{t+1} \tag{6}$$

and that the price of a bond with maturity n is an affine function of the state X_t plus a maturity specific disturbance v_t^n

$$p_t^n = A_n + B_n' X_t + v_t^n. \tag{7}$$

That is, bond prices depend on the exogenous factors x_t as well as on the average higher order expectations of these factors. The maturity specific disturbances v_t^n prevents bond prices from revealing the expectations of other traders. The maturity specific disturbances thus plays a similar role as the noise traders in Admati (1985) and can be motivated as in Duffee (2011) as arising from "*preferred habitats, special repurchase rates, or variations in liquidity*".

It is perhaps worth pointing out here that even though the state vector is high dimensional, this by itself does not increase our degrees of freedom in terms of fitting bond yields. The fact that the endogenous state variables $x_t^{(k)}$ are rational expectations of the lower order expectations in $x_t^{(k-1)}$ disciplines the law of motion (6) and the matrices $\mathbf{M}$ and $\mathbf{N}$ are completely pinned down by the parameters of the process governing the true exogenous factors x_t and how precise traders' signals about x_t are. How this is done in practice is described in the Appendix.

B. The average expectations operator H

For future reference, it is useful to define the average expectations operator $H : \mathbb{R}^{\bar{k}+1} \rightarrow \mathbb{R}^{\bar{k}+1}$ that takes a hierarchy of expectations and moves all expectations one step up in orders of expectations by annihilating the first element in a hierarchy of expectations and sets all expectations of order $k > \overline{k}$ equal to zero so that

$$\begin{bmatrix} x_t^{(1)} \\ \vdots \\ x_t^{(\bar{k})} \\ 0 \end{bmatrix} = H \begin{bmatrix} x_t \\ x_t^{(1)} \\ \vdots \\ x_t^{(\bar{k})} \end{bmatrix} \tag{8}$$

The matrix H will be useful for deriving expressions that depend on expectations about the aggregate state X_t and expectations about other traders' expectations.

C. The stochastic discount factor of trader j

Above, we conjectured a law of motion for the state X_t and how bond prices depend on the state. Traders want to be compensated for the risk associated with holding bonds and this risk arises from two sources, i.e. from uncertainty about future states X_t and from future realizations of the maturity specific shocks v_t^n. Here we specify the functional form for traders' stochastic discount factors that will be used to price this risk.

The stochastic discount factor of trader j is denoted M_{t+1}^j and in the absence of arbitrage, the relationship

$$p_t^{n+1} = \log E\left[M_{t+1}^j P_{t+1}^n | \Omega_t^j\right] \tag{9}$$

must be satisfied for each trader j and maturity n. With heterogeneous information, expectations about future prices and discount factors may differ across traders but in the absence of arbitrage opportunities, any expected return in excess of the risk-free rate must in equilibrium be compensation for risk. The compensation for a given quantity of risk required by trader j must thus be determined partly by the same trader-specific factors that determine return expectations. Following the full information affine asset pricing literature, we specify the logarithm of trader j's SDF to follow

$$m_{t+1}^j = -r_t - \frac{1}{2}\Lambda_t^{j\prime}\Sigma_a\Lambda_t^j - \Lambda_t^{j\prime}\mathbf{a}_{t+1}^j. \tag{10}$$

The vector $\mathbf{a}_{t+1}^j$ is the period $t+1$ innovation to trader j's state vector X_t^j, conditional on information available to trader j up to period t defined as

$$\mathbf{a}_{t+1}^j \equiv X_{t+1}^j - E(X_{t+1}^j | \Omega_t(j)) \tag{11}$$

with conditional covariance matrix Σ_a. That is, the innovation vector $\mathbf{a}_{t+1}^j$ spans the risk that trader j requires compensation for. The state X_t^j consists of a vector of trader j specific exogenous factors x_t^j given by

$$x_t^j = x_t + Q\eta_t^j : \eta_t^j \sim N(0, I) \tag{12}$$

as well as trader j's (up to order $\bar{k}$) expectation of the common latent d-dimensional vector x_t so that

$$X_t^j \equiv \begin{bmatrix} x_t^j \\ \begin{bmatrix} I_{\bar{k}\times d} & \mathbf{0} \end{bmatrix} E\left[X_t \mid \Omega_t^j\right] \end{bmatrix}. \tag{13}$$

The vector x_t^j is also the source of trader j's trader-specific information about the unobservable exogenous state x_t. The precision of the signals x_t^j are common across traders and determined by the matrix Q. The individual vector of risk prices Λ_t^j in the stochastic discount factor (10) is an affine function of trader j's state X_t^j

$$\Lambda_t^j = \Lambda_0 + \Lambda_X X_t^j \tag{14}$$

Trader j observes x_t^j but cannot by direct observation disentangle the common factors x_t from the idiosyncratic component $Q\eta_t^j$. Even though the idiosyncratic shocks η_t^j are purely transitory they will in general have persistent effects on trader j's (higher order) expectations so that the difference between trader j's state X_t^j and the average state X_t can be expressed as

$$X_t^j - X_t = \mathcal{Q}(L)\eta_t^j \tag{15}$$

where $\mathcal{Q}(L)$ is an infinite order lag polynomial. If $Q = 0$ so that $x_t^j = x_t$, the vector $\mathbf{a}_{t+1}^j$ would simply be the innovation to the exogenous common factors implying that $\mathbf{a}_{t+1}^j = C\varepsilon_{t+1}$ and $\Sigma_a = CC'$ in (4). The model then collapses to the full information affine model since the trader-specific state x_t^j then perfectly reveals the true (common) latent factors x_t and all traders require the same compensation for risk. The history of realizations of the idiosyncratic components $Q\eta_t^j$ also determines how trader j's expectations about future returns differ from the average expectation, since the only source of heterogeneous information is the observation of x_t^j.

D. Traders' filtering problem

The SDF introduced above prices conditional uncertainty about future bond prices. This uncertainty depends both on the uncertainty inherent in the unknowable nature of the future as well as on the uncertainty arising from the fact that the current state of the world is not directly observable. In order to form expectations about future bond prices, traders need to form an estimate of the current aggregate state X_t. Since the model is linear with Gaussian shocks, the Kalman filter delivers optimal state estimates.

Traders know the law of motion of the state X_t as well as how the state maps into the vector of observable variables. They use this knowledge to form model consistent expectations of the aggregate state. In each period traders observe the short rate r_t, a vector of current bond yields

$$\mathbf{y}_t \equiv \begin{bmatrix} \frac{1}{2}p_t^2 & \cdots & n^{-1}p_t^n & \cdots & N^{-1}p_t^N \end{bmatrix}' \tag{16}$$

as well as the trader-specific factors x_t^j. The variables observable to trader j can be collected in the vector z_t^j defined as

$$z_t^j \equiv \begin{bmatrix} x_t^j \\ r_t \\ \mathbf{y}_t \end{bmatrix} \tag{17}$$

Through observing equilibrium bond yields, traders can extract information about the unobservable state of the economy. This contrasts with difference-in-beliefs models where agents "agree to disagree". When traders agree to disagree, the beliefs of all agents are common knowledge and from the traders' perspective, there is no additional information contained in endogenous prices.

Traders do not forget and the information set of trader j is the filtration defined by

$$\Omega_t^j = \left\{ z_t^j, \Omega_{t-1}^j \right\} \tag{18}$$

The law of motion of the state (6) and the definition of the observables (17) then let us describe the filtering problem of trader j as a standard state space system

$$X_{t+1} = \mu + \mathbf{M}X_t + \mathbf{N}\mathbf{u}_{t+1} \tag{19}$$

$$z_t^j = \mu_z + DX_t + R \begin{bmatrix} \mathbf{u}_t \\ \eta_t^j \end{bmatrix} \tag{20}$$

where $\mathbf{u}_{t+1}$ is a vector containing the innovations to the exogenous state ε_{t+1} and the vector of maturity specific shocks $\mathbf{v}_{t+1}$. The matrices D and R are defined in the Appendix. Trader j's state estimate evolves according to the Kalman filter updating equation

$$E\left[X_t \mid \Omega_t^j\right] = (I - KD)\,\mathbf{M}E\left[X_{t-1} \mid \Omega_{t-1}^j\right] + Kz_t^j \tag{21}$$

where K is the Kalman gain. Since bond yields are part of the observation vector z_t^j, the selector matrix D in the measurement equation (20) is partly a function of the vectors B_n in the bond price equation (7). This implies that we have to solve simultaneously for the filtering problem and the pricing equation (7).

E. Higher order expectations and bond prices

Individual traders are assumed to be price takers. In practice, this means that when we evaluate the no-arbitrage condition (9) for trader j, we replace the expectation of the next period price P_{t+1}^n by trader j's expectation of what other traders will be willing to pay for the bond in the next period, rather than what he thinks he would be willing to pay for it himself. What other traders will be willing to pay in period $t+1$ depends on their stochastic discount factors and their expectations in period $t+1$ of the price of the bond in period $t+2$. In equilibrium, trader j could base his prediction of the bond price in period $t+1$ on his prediction of what *any* other trader $i \neq j$ would be willing to pay for the bond. However, since trader j does not have any information relevant for predicting the expectations and stochastic discount factors of any other particular trader, trader

j's expectation of some other trader i's expectations coincides with trader j's expectation about the average expectation, that is

$$E\left[P^n_{t+1} \mid \Omega^j_t\right] = E\left[E\left[M^{i\neq j}_{t+2}P^{n-1}_{t+2}|\Omega^{i\neq j}_{t+1}\right] \mid \Omega^j_t\right] \qquad (22)$$
$$= E\left[\int E\left[M^i_{t+2}P^{n-1}_{t+2}|\Omega^i_{t+1}\right] di \mid \Omega^j_t\right]$$

Trader j's no-arbitrage condition can thus be written as

$$P^n_t = E\left[M^j_{t+1}\left(\int E\left[M^i_{t+2}P^{n-1}_{t+2}|\Omega^i_{t+1}\right] di\right) \mid \Omega^j_t\right] \qquad (23)$$

Continued substitution of average expectations about future bond prices in a similar fashion makes it possible to write the n period bond price as a product of trader j's n (higher order) expectations of average future stochastic discount factors

$$P^n_t = E\left[M^j_{t+1}\left(\int E\left[M^i_{t+2} \quad \cdots \quad \int E\left[M^{i'}_{t+n} \mid \Omega^{i'}_{t+n-1}\right] di' \quad \cdots \quad |\Omega^i_{t+1}\right] di\right) \mid \Omega^j_t\right] \qquad (24)$$

With heterogeneous information sets, this product will generally differ from the product of trader j's expectations about his own stochastic discount factor over the next n periods and it is because of this fact that speculative behavior arises.[4]

The no-arbitrage condition (9) has to hold for all traders at all times. This implies that in equilibrium, we could choose any trader j's state X^j_t as being the state variable that bond prices are a function of. However, the most convenient choice from a modeling perspective is to let bonds be priced by the SDF of the fictional trader whose state X_t is defined to coincide with the cross-sectional average state so that

$$X_t \equiv \int X^i_t di \qquad (25)$$

The identity of the average trader will change over time as idiosyncratic shocks change an individual trader's relative position in the cross-sectional distribution. However, the identity of the average trader is of no consequence and the advantage of letting the average trader's SDF price bonds is that it allows us to write log bond prices in the conjectured form (7), i.e. as a function of the aggregate state X_t. In the Appendix we show that combining the log of the expression (24) for the average trader defined as (25) and the functional form of the log of the stochastic discount factor

[4] Readers interested in an in-depth discussion and more formal results on the properties of higher order expectations can consult Allen, Morris and Shin (2006) and Nimark (2012).

(10) results in recursions for the matrices A_{n+1} and B_{n+1} in the conjectured bond price equation (7) that are given by

$$\begin{aligned} A_{n+1} &= -\delta_0 + A_n + B_n'\mu^P + \frac{1}{2}\left[B_n'\Sigma_{t+1|t}B_n' + V_nV_n'\right] \\ &\quad -B_n'\Sigma_{t+1|t}\Lambda_0 + B_n'\mathbf{N}V_n' - V_n\mathbf{N}'\Lambda_0 \end{aligned} \tag{26}$$

and

$$B_{n+1}' = -\delta_X' + B_n'\mathbf{M}H - (B_n'\Sigma_{t+1|t} + V_n\mathbf{N}')\Lambda_X \tag{27}$$

where H is the average expectations operator (8). The matrix $\Sigma_{t+1|t}$ is the conditional covariance of the errors of traders' one step ahead expectations of the state, using the notation

$$\Sigma_{t+s|t} \equiv E\left(X_{t+s} - E\left[X_{t+s} \mid \Omega_t^j\right]\right)\left(X_{t+s} - E\left[X_{t+s} \mid \Omega_t^j\right]\right)'. \tag{28}$$

As in a full information set up, the recursions (26) and (27) can be started from

$$A_1 = -\delta_0 \tag{29}$$

$$B_1 = \begin{bmatrix} -\delta_X' & \mathbf{0} \end{bmatrix} \tag{30}$$

since $p_t^1 = -r_t$.

The recursive expressions above are similar to the corresponding expressions under full information. Where there are differences, those arise because of the fact that under imperfect information, the conditional covariance of the state $\Sigma_{t+1|t}$ now depends not only on the state innovations in the next period, i.e. the vector $\mathbf{N}\mathbf{u}_t$ in the law of motion (6), but also on the uncertainty about the current state $\Sigma_{t|t}$. In addition, the maturity specific disturbances v_t^n affect the state X_t directly when traders are imperfectly informed. The reason is that the shocks v_t^n affect bond yields and traders use the information in bond yields to form an estimate of the state, and the state X_t is partly made up of traders' estimates of the state. Under full information, the maturity specific shocks do not affect traders' estimates of the state since these are by construction equal to the true (exogenous) state x_t. The term $V_n\mathbf{N}'$ then equals zero, even though the maturity specific shocks still contribute to the overall variance of bond yields and traders still gets compensated for this risk through the term V_nV_n' in equation (26) of the bond price recursions.

F. Expected returns and trader j's price of risk

We now have all the components in place to show how the filtering problem of the agents together with the conjectured price process and the stochastic discount factor relate the expected excess return conditional on trader j's information set to the compensation of risk as a function of trader j's state X_t^j. Since r_t and p_t^{n+1} are observable we can express the expected excess return conditional on trader j's information as

$$E\left[p_{t+1}^n \mid \Omega_t^j\right] - p_t^{n+1} - r_t = A_n - A_{n+1} - \delta_0 + (B_n' \Sigma_{t+1|t} + V_n \mathbf{N}')\Lambda_X X_t^j \tag{31}$$

where we used that

$$E\left[p_{t+1}^n \mid \Omega_t^j\right] = A_n + B_n' \mathbf{M} H X_t^j \tag{32}$$

and the expression for B_{n+1} from the recursion (27). As in the standard model, a positive expected excess return is compensation for risk. The constant component of this compensation, i.e. $A_n - A_{n+1} - \delta_0$ is a function of the constant price of risk Λ_0 just as in the standard model. The difference here is that the time-varying component of expected excess returns is partly trader-specific. In the absence of arbitrage, it must be the case that a trader who expects a higher excess return than other traders must also require more compensation for risk. The trader-specific price of risk must therefore be a function of the same trader-specific state that determines a trader's expectations. In equation (31) trader j's compensation for risk is the last term on the right hand side, which is a function of the conditional variance $(B_n' \Sigma_{t+1|t} + V_n \mathbf{N}')$ and the trader js hierarchy of expectations X_t^j. The matrix Λ_X determines how they are combined into the time varying component of trader $j's$ required compensation for risk in period t.

This ends the presentation of the model. We now turn to how it can be used to decompose bond prices.

II. Decomposing the yield curve

This section demonstrates how the term structure can be decomposed into the standard components consisting of expected future short rates and risk premia as well as speculative components driven by traders exploiting what they perceive to be inaccurate average expectations about future bond yields. Harrison and Kreps (1978) defined speculative behavior as when the possibility of reselling an asset before it matures changes its equilibrium price. In the model presented here, we implicitly assume that long maturity bonds are traded in every period and that traders therefore need to predict what other traders will be willing to pay for a bond at the next trading opportunity. Since the price other traders will be willing to pay for a bond in the future depends on their future risk adjusted expectations of bond prices further into the future, individual traders will need to form higher order expectations about all the determinants of future bond prices. That is, traders need to form expectations about other traders' expectations about both future risk premia as well as future short rates.[5]

A. The speculative component in bond prices

It is well known that higher order expectations are distinct from first order expectations when traders have access to heterogeneous information (e.g. Allen, Morris and Shin 2006, Bacchetta and van Wincoop 2006 and Nimark 2012). When individual traders believe that average expectations of the current state are inaccurate, that is, when first and higher order estimates of the current state do not coincide, individual traders can predict the direction that other traders will adjust their expectations in the future. To see why, note that if an individual trader in period t thinks that other traders have an inaccurate estimate of the current state and therefore inaccurate expectations about bond yields in period $t+n$, it is rational for the individual trader to expect other traders to revise their expectations when more information becomes available in the future. The individual trader thus expects other rational traders to revise their expectations between period t and $t+n$ towards what the individual trader believes is a more accurate expectation of bond yields in period $t+n$. When they take advantage of this predictability of other traders' forecast revisions, speculative dynamics arise. If by chance an individual trader thinks that the average estimate of the current state is the same as his own best estimate, no such predictability of others forecasts revisions is possible. The forecast of others' forecasts then coincide with the individual's own forecast, which follows a martingale.

This reasoning suggest that we should define the speculative component in an n period bond price as the difference between the actual price of the bond, and the counterfactual price that would prevail if average first and higher order expectations coincided. This counterfactual price is what

[5] Relative to the model of Nimark (2012) where the focus is on heterogenous expectations about future risk-free short rates there is thus an additional speculative component to the term structure here driven by traders exploiting what they perceive to be inaccurate market expectations about future risk premia.

Allen, Morris and Shin (2006) refers to as the "consensus value" of an asset and the difference between the "consensus value" and the actual price is what Bacchetta and van Wincoop (2006) refers to as the "higher order wedge".

In our framework, the consensus value of an n period bond denoted $\overline{p}_t^n$ is defined as the price that would prevail if by chance all average higher order expectations coincided with the average first order expectation, i.e.

$$x_t^{(1)} = x_t^{(k)} : k = 1, 2, 3, ... \tag{33}$$

The consensus value of an n period bond is then given by

$$\overline{p}_t^n = A_n + B_n^{'} \overline{H} X_t + v_t^n \tag{34}$$

where the matrix $\overline{H}$ is what we may call the *consensus operator* $\overline{H} : \mathbb{R}^{\bar{k}+1} \to \mathbb{R}^{\bar{k}+1}$ defined so that

$$\begin{bmatrix} x_t \\ x_t^{(1)} \\ \vdots \\ x_t^{(1)} \end{bmatrix} = \overline{H} \begin{bmatrix} x_t \\ x_t^{(1)} \\ \vdots \\ x_t^{(\bar{k})} \end{bmatrix} \tag{35}$$

That is, $\overline{H}$ is a matrix that takes a hierarchy of expectations and equates average higher order expectations with the average first order expectation. We can use $\overline{H}$ to decompose the current n period bond price into a component that depends on the average first order expectation and a speculative component that is the difference between the actual price and the counterfactual consensus value $\overline{p}_t^n$ as follows

$$p_t^n = A_n + B_n^{'} \overline{H} X_t + \underbrace{B_n^{'} \left(I - \overline{H} \right) X_t}_{\textit{total speculative term}} + v_t^n \tag{36}$$

The speculative term is thus the difference between the actual price and the answer you would get if you asked the "average" trader what he thinks the price would be if all traders, by chance, had the same state estimate as he did (while holding conditional uncertainty constant).

B. Decomposing bond prices

The speculative term (36) can be further decomposed into an (even!) more interesting form in order to separate speculation related to expected future short rates from speculation about future risk premia. First, note that by repeated substitution in the bond price recursions (27), the vector B_n can be expressed as

$$B_n' = -\sum_{s=0}^{n-1} \delta_X \left[\mathbf{M}H - \Sigma_{t+1|t}\Lambda_X\right]^s + \sum_{s=1}^{n-1} V_s \mathbf{N}\Lambda_X \left[\mathbf{M}H - \Sigma_{t+1|t}\Lambda_X\right]^{n-1-s} \tag{37}$$

When expanded, the expression for B_n' contains the term $-\sum_{s=0}^{n-1} \delta_X \left[\mathbf{M}H\right]^s$. That is, the current price of an n period bond partly depends on the cumulative sum of higher order short rate expectations between period t and $t+n-1$. We can thus decompose B_n' into a term consisting of (higher order) expectations about the short rate r_t and a term W_n capturing current and (higher order) expectations about future risk premia.

$$B_n' = -\delta_X \sum_{s=0}^{n-1} \left[\mathbf{M}H\right]^s + W_n \tag{38}$$

That is, W_n contains the sum of the remaining terms in (37) that involve Λ_X. Using $\overline{H}$ and W_n we can write the actual bond price as

$$\begin{aligned} p_t^n \quad = \quad & \underbrace{\widetilde{A}_n + W_n \overline{H} X_t}_{common\ risk\ premia} \\ & - \underbrace{\sum_{s=0}^{n-1} \int E\left(r_{t+s} \mid \Omega_t^j\right) dj}_{average\ expectations\ of\ future\ short\ rates} \\ & \underbrace{-\delta_X \sum_{s=0}^{n-1} \left[\mathbf{M}H\right]^s \left(I - \overline{H}\right) X_t}_{short\ rate\ speculation} \\ & + \underbrace{W_n \left(I - \overline{H}\right) X_t}_{risk\ premia\ speculation} \\ & + v_t^n \end{aligned} \tag{39}$$

where the cross-sectional average first order expectations about future short rates can be computed as

$$\int E\left(r_{t+s} \mid \Omega_t^j\right) dj = \delta_0 + \delta_X \mu^X + \delta_X \left[\mathbf{M}H\right]^s \overline{H} \left(X_t - \mu^X\right) \tag{40}$$

The first line is the common component of risk premia made up of the term $\widetilde{A}_n$ that contains the unconditional mean of p_t^n minus the cumulative sum of the unconditional mean of risk-free rates

$$\widetilde{A}_n \equiv A_n - n\left(\delta_0 + \delta_X \mu^X\right) - \delta_X \sum_{s=0}^{n-1} \left[\mathbf{M}H\right]^s \mu^X \tag{41}$$

The time-varying common component of risk premia $W_n\overline{H}X_t$ contains the cumulative sum of discounted risk adjusted average first order expectation of future risk premia. The second line is the cumulative sum of the cross-sectional average expectation of future short rates. The first two lines of (39) thus corresponds to the classic terms of the yield curve decomposition in Cochrane and Piazzesi (2008) and Joslin, Singleton and Zhu (2011) and are independent of higher order expectations.

The term on the third line is the cumulative sum of the difference between higher order and average first order expectations about future short rates. Similarly, the fourth line contains the difference between average first order expectations and the higher order expectations about future risk premia. The difference between higher and first order expectations can equivalently be thought of as higher order prediction errors, i.e. expectations about the expectations errors other traders are making, since traders believe their first order expectations to be optimal estimates of the true state. The last two terms in (39) are thus capturing the speculative part of bond prices due to traders taking advantage of what they perceive to be mistakes in other traders' expectations about future short rates and risk premia. As proved more formally in Nimark (2012), this type of speculative component must be orthogonal to public information in real time since individual traders cannot predict the errors other traders are making based on information available to all traders. This makes the speculative component statistically distinct from the other terms in the decomposition and has implications for how the speculative terms can be estimated. In the next section we discuss the empirical specification, how the model can be estimated and the speculative terms quantified.

III. Empirical specification

In order to make the model presented above operational we will need to be specific about some of the details that up until this point have been presented at a more general level. We start by discussing how the factor processes are normalized and some related identification issues. We then turn to how the prices of risk can be parameterized parsimoniously when higher order expectations enter as state variables. We also describe how the cross-sectional dimension of the Survey of Professional Forecasters can be exploited to make sharper inference about the precision of information available to traders, and more generally, how to estimate the parameters of the model using likelihood based methods.

A. Exogenous factor dynamics

The first choice we need to make is to decide how many factors to include in the exogenous vector x_t. In the benchmark specification, x_t is a three dimensional vector so that in the special case with perfectly informed traders, the model collapses to a standard three factor affine Gaussian no-arbitrage model. The factors x_t follow a VAR(1) process under the physical measure

$$x_{t+1} = \mu^P + F^P x_t + \mathbf{C}\varepsilon_{t+1}. \tag{42}$$

Since the factors are latent we need to impose restrictions on their laws of motion and on how those factors relate to the short rate. We follow Joslin, Singleton and Zhu (2011) and restrict the dynamics of the factors under the risk neutral measure to also follow VAR(1) dynamics

$$x_{t+1} = \mu^Q + F^Q x_t + \mathbf{C}\varepsilon^Q_{t+1} \tag{43}$$

with the restrictions that $\mu^Q = 0$ and that the matrix $\mathbf{F}^Q$ is diagonal with the factors ordered in descending degree of persistence under the risk neutral dynamics. Furthermore, C is restricted to be lower triangular. Finally, the vector δ'_X of the short rate equation (3) is restricted to have its first three elements equal to one and all other elements equal to zero. This ensures that all parameters are identified in the special case of perfectly informed traders.

B. Trader's information sets

Trader j observes the factors x_j as defined in (12) which is the source of trader j's heterogeneous information about the common factors x_t. Each trader also observes the risk-free short rate r_t. In addition to these exogenous signals, all traders can observe all bond yields up to maturity N, where N is the largest maturity used in the estimation of the model. Here, the longest maturity yield that we will use in estimation is a 10 year bond implying that $N = 40$ with quarterly data. That traders observe bond yields of all maturities is of no particular importance and there is little information added by observing additional bond yields as long as the yields of at least three different maturities (more or less evenly spread out over the yield curve) are observed.

C. Prices of risk parametrization

The traders in the model require compensation for the risk associated with the conditional variance of future bond prices. As explained in Section 2 above, expected excess returns are partly trader-specific but must in the absence of arbitrage be compensation for risk. The required compensation for risk must therefore also partly be trader-specific and trader $j's$ required compensation for risk is a function of the same variables that the expected excess returns are conditioned on. Following the full information affine literature as closely as possible, we specify the vector of risk prices of trader j as an affine function of trader j's state X_t^j

$$\Lambda_t^j = \Lambda_0 + \Lambda_X X_t^j \tag{44}$$

The state vector X_t^j is high dimensional and as a consequence, leaving Λ_0 and Λ_X completely unrestricted would result in a very large number of free parameters. In order to avoid an over-parameterized model we therefore restrict Λ_0 and Λ_X as follows

$$\begin{aligned}\Lambda_t^j &= \begin{bmatrix} \lambda_0 \\ \mathbf{0}_{(3\bar{k}-3)} \end{bmatrix} + \begin{bmatrix} \lambda_1 & \mathbf{0}_{3\times(3\bar{k}-3)} \\ \mathbf{0}_{(3\bar{k}-3)\times 3} & \mathbf{0}_{(3\bar{k}-3)\times(3\bar{k}-3)} \end{bmatrix} X_t^j \\ &+ \begin{bmatrix} \lambda_2\mathbf{I}_3 & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \cdots & \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times3} & \alpha\lambda_2\mathbf{I}_3 & \mathbf{0}_{3\times3} & \ddots & \vdots \\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \alpha^2\lambda_2\mathbf{I}_3 & \cdots & \mathbf{0}_{3\times3} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \mathbf{0}_{3\times3} & \cdots & \mathbf{0}_{3\times3} & \cdots & \alpha^{\bar{k}}\lambda_2\mathbf{I}_3 \end{bmatrix} (I-H)X_t^j \end{aligned} \tag{45}$$

The sum of the two matrices multiplying X_t^j in (45) then equals the matrix Λ_X in (44). The matrix $(I - H)$ takes the difference between the expectations of the state and the true state, i.e.

$$(I-H)X_t^j = \begin{bmatrix} x_t - E\left[x_t \mid \Omega_t^j\right] \\ E\left[x_t \mid \Omega_t^j\right] - E\left[x_t^{(1)} \mid \Omega_t^j\right] \\ \vdots \\ E\left[x_t^{(\bar{k}-1)} \mid \Omega_t^j\right] - E\left[x_t^{(\bar{k})} \mid \Omega_t^j\right] \end{bmatrix} \tag{46}$$

and the term $(I - H)X_t^j$ thus equals zero in the special case when traders are perfectly informed. The three dimensional vector λ_0 and the 3×3 matrix λ_1 in the first row of (45) thus completely specifies the price of risk in the perfect information case, just as it would in the standard full information affine three factor model. It then follows that the price of risk associated with traders taking advantage of what they perceive to be inaccurate market expectations is governed by the scalars λ_2 and α. These are the only two parameters added in the specification of the price of risk relative to the full information set up.

The parameter λ_2 controls how the price of risk depends on differences in higher order expectations. To understand why the differences between adjacent orders of expectations in (46) should affect the required compensation for risk, recall that when traders have access to heterogeneous information they can predict the average expectation errors of other traders. When say, a trader's first and second order expectations differ, that must imply that the individual trader thinks that other traders' expectations are inaccurate since he thinks his own first order expectation is optimal. The excess return that can be earned by the individual trader from exploiting this difference must in the absence of arbitrage be compensation for risk.[6] The parameter λ_2 determines just by how much this compensation varies with the difference between different orders of expectations. The parameter α in turn regulates how differences in beliefs between different orders of expectations depends on which orders are involved. For example, if $\alpha = 1$ then the difference between the first and the second order expectation price risk the same way that the difference between second and third order, and so on. If $\alpha < 1$ then differences between higher order expectations have a smaller impact on the price of risk than differences between lower order expectations.

The matrix that prices risk as a function of differences in higher order expectations is thus parameterized by just two free parameters. We experimented with more flexible specifications, but found that restricting Λ_X in this way did not appear to restrict the dynamics of the speculative component in bond prices. This is probably due to the fact that the vector of higher order prediction errors $(I - H)X_t^j$ in practice appears to be well-described by a one factor process.[7]

D. Choosing the maximum order of expectation $\overline{k}$

In Nimark (2011) it is demonstrated that, under quite general conditions, it is possible to accurately represent the dynamics of a model with heterogeneously informed agents by a finite dimensional state vector, in spite of the infinite regress of "forecasting the forecasts of others" that naturally arises in such models when agents need to consider events over an infinite horizon. Since we only consider bonds with a finite maturity, the present model does not give rise to an *infinite* regress of higher order expectations. Still, rather than letting the N orders of expectations about

[6] In Nimark (2012), the expected excess return due to heterogeneous information is compensation for the risk associated with holding a less balanced portfolio, i.e. holding more of the risky bond that the trader thinks will yield an excess return. Though portfolio decisions are not modeled explicitly here, the framework is consistent with such an interpretation.

[7] The virtual one factor structure of $(I - H)X_t^j$ can be verified numerically on a case-by-case basis by inspecting the scree plot of the matrix $(I - H)\Sigma_X^j(I - H)'$ where $\Sigma_X^j \equiv E\left[X_t^j X_t^{j\prime}\right]$. We think that the apparent one factor structure of the speculative component is a result of having just enough shocks to make equilibrium bond prices not perfectly revealing. Including one aggregate shock less in the model would make bond prices perfectly revealing and the speculative dynamics would in that case become a "zero-factor" process, i.e. be identically zero at all times.

future SDFs (that would be required to compute the expression (24) for the maximum maturity N) be the state variables of the model, it is more convenient to use a state representation of the form (5), i.e. to let the state consist of the higher order expectations about the current exogenous factors x_t. While the results in Nimark (2011) show that the equilibrium can be approximated to arbitrary accuracy with a finite dimensional state, what finite means in practise has to be checked on a case by case basis. For all parametrizations we considered, the equilibrium dynamics converge rapidly as the maximum order of expectation is increased. In the final specification used for estimation, we set $\overline{k} = 15$, which is more than sufficient as most of the dynamics are captured by the first five or six orders of expectations.

We model traders as explicitly forming higher order expectations, i.e. expectations about other traders' expectations and the equilibrium representation can be taken as a literal description of traders' behavior. Given the prevalence of quotes of Keynes' beauty contest metaphor in the finance literature it appears that many people find the related intuition appealing. However, it may strain credulity to think that traders form expectations beyond two or three orders and here we solve the model by assuming that traders form up to the 15^{th} order of expectations. However, an alternative interpretation is to view the equilibrium representation simply as a convenient recursive functional form to model agents who have access to heterogeneous information and condition on the entire history of observables in order to predict next period bond yields. The main advantage of the method is then to deliver a (relatively) low dimensional and time invariant representation of equilibrium dynamics.

E. Estimating the model using bond yields and survey data

The parameters of the model can be estimated using likelihood based methods. We use quarterly data on bond yields of maturity 1,2,3,4,5,6,7,8,9 and 10 years with the sample spanning the period 1971:Q4 to 2011:Q4. The zero-coupon yield data is taken from the Gurkaynak, Sack and Wright (2007) data set. In addition to bond yields we also use one quarter ahead forecasts of the T-Bill rate and the 1 quarter ahead forecasts of the 10 year bond rate from the Survey of Professional Forecasters (SPF). In the model, the cross-sectional distribution of traders' one period ahead forecasts of the short rate is Gaussian with mean and variance given by

$$E\left[r_{t+1} \mid \Omega_t^j\right] \sim \mathbf{N}\left(A_1 + B_1\mathbf{M}HX_t, B_1\mathbf{M}\Sigma_j\mathbf{M}'B_1'\right) \tag{47}$$

and that of the 10 year yield is

$$E\left[y_{t+1}^{40} \mid \Omega_t^j\right] \sim \mathbf{N}\left(\frac{1}{40}A_{40} + \frac{1}{40}B_{40}\mathbf{M}HX_t, \frac{1}{40}B_{40}'\mathbf{M}\Sigma_j\mathbf{M}'B_{40}\frac{1}{40}\right) \tag{48}$$

where Σ_j is the cross-sectional covariance of expectations about the current state, i.e.

$$\Sigma_j \equiv EH\left(X_t^j - X_t\right)\left(X_t^j - X_t\right)' H' \tag{49}$$

As econometricians, we can thus treat the individual survey responses of T-Bill rate forecasts as noisy measures of the average expectation of the short rate r_t where the variance of the "noise" is determined by the cross-sectional model implied variance of short rate expectations. The Appendix contains details of how to compute the cross-sectional variance Σ_j in practice. The deviations of individual traders' forecasts from the average forecasts are caused by idiosyncratic shocks that are normally distributed and independent across traders. The covariance of the measurement errors can thus be specified as the scalars $B_1'\mathbf{M}\Sigma_j\mathbf{M}'B_1$ and $\frac{1}{40}B_{40}'\mathbf{M}\Sigma_j\mathbf{M}'B_{40}\frac{1}{40}$ times an identity matrix.

We now have all the ingredients needed to evaluate the log likelihood function

$$\log L\left(\overline{Z}^T\right) = -\frac{1}{2}\left\{\sum_{t=1}^{T} 2\pi \dim(\widetilde{Z}_t) + \log\left|\Sigma_{t|t-1}\right| + \widetilde{Z}_t\Sigma_{t|t-1}^{-1}\widetilde{Z}_t\right\} \tag{50}$$

conditional on the sample $\overline{Z}^T$ by computing the innovations

$$\widetilde{Z}_t = \overline{Z}_t - E\left[\overline{Z}_t \mid \overline{Z}^{t-1}\right] \tag{51}$$

from the state space system

$$X_t = \mu^X + \mathbf{M}X_{t-1} + \mathbf{N}\mathbf{u}_t \tag{52}$$

$$\overline{Z}_t = \mathbf{d}_t + \overline{D}_t X_t + \overline{R}_t\varepsilon_t : \varepsilon_t \sim N(0, I) \tag{53}$$

where

$$\mathbf{d}_t = \begin{bmatrix} -\frac{1}{4}A_4 \\ -\frac{1}{8}A_8 \\ \vdots \\ -\frac{1}{40}A_{40} \\ \frac{1}{40}\mathbf{1}_{(m\times 1)} \times A_1 \end{bmatrix}, \quad \bar{D}_t = \begin{bmatrix} -\frac{1}{4}B_4' \\ -\frac{1}{8}B_8' \\ \vdots \\ -\frac{1}{40}B_{40}' \\ \mathbf{1}_{(m\times 1)} \times B_1'\mathbf{M}H \\ \frac{1}{40}\mathbf{1}_{(m\times 1)} \times B_{40}'\mathbf{M}H \end{bmatrix} \tag{54}$$

$$\overline{R}_t\overline{R}_t' = \begin{bmatrix} RR' & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I_m \times B_1'\mathbf{M}\Sigma_j\mathbf{M}'B_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_m \times \frac{1}{40}B_{40}'\mathbf{M}\Sigma_j\mathbf{M}'B_{40}\frac{1}{40} \end{bmatrix}$$

and m is the number of survey responses available in period t. The number of survey responses vary over time and surveys are not available at all for the period before 1981:Q3. The dimensions of $\mathbf{d}_t, \bar{D}_t$ and $\bar{R}_t$ are thus varying over time as well. This presents no particular problem, but may influence the precision of our estimates of the state, i.e. we will have more precise estimate of the latent state X_t when there is a large number of survey responses available. Using individual survey responses and likelihood based methods also naturally incorporates that we have more precise information about the expectations of traders when there are 50 responses (the sample maximum) compared to when there are only 9 responses (the sample minimum). This information is lost when using measures of central tendency like a mean or median forecast.

Using individual survey responses also allow us to exploit the second moment of surveys, i.e. the cross-sectional variance of surveys, to inform the posterior parameter estimates. Clearly, not all parameter values will imply a cross-sectional distribution of expectations among the traders populating the model that is consistent with the cross-sectional variance observed in the surveys. The cross-sectional dispersion of expectations is a non-monotonic function of the precision of the trader-specific signals. When the trader-specific signals are very precise, the cross-sectional dispersion is close to zero and the dynamics of bond yields will be close to those of the full information model. When the trader-specific signals are very imprecise, traders attach little weight to them and again, the cross-sectional dispersion would be close to zero. To match the substantial dispersion observed in surveys, the model requires intermediate values for the precision of the trader-specific signals.

Less obviously, the cross-sectional dispersion in surveys will also discipline the dynamics of bond prices more generally. If bond prices are too revealing about the latent exogenous factors x_t, the cross-sectional dispersion will be too low relative to the dispersion in the survey data, regardless of the precision of the trader-specific signals. Through this channel, the model together with the individual survey responses imposes joint restrictions on the dynamics of bond prices and the cross-sectional dispersion of expectations.

An alternative strategy to use the survey data is to treat individual responses as noisy measures of a common expectation held by all traders (see Kim and Orphanides (2005) and Chernov and Mueller (2012). Others have used a measure of central tendency from the surveys, like a mean or median, to represent a noisy measure of the expectations of a representative agent (see Piazzesi and Schneider (2011)). The difference between this approach and ours is that these alternative strategies do not impose any joint restrictions on the dynamics of bond prices and the cross-sectional dispersion of survey responses since the standard affine model is silent on these.

IV. Empirical results

We can now present the main empirical results of the paper. The model is parameterized by the matrices A^Q and C which governs the processes of the common latent factors x_t, the matrix Q which specifies the standard deviation of the idiosyncratic noise in the trader-specific signals about x_t, the scalar δ_0 which is the unconditional mean of the short risk-free rate r_t, σ_v the standard deviation of the maturity specific disturbances v_t^n (specified so that $\sqrt{var\left(v_t^n\right)} = n\sigma_v$, i.e. so that the direct impact of the shocks on *yields* are constant across maturities) and the vector μ^P, matrix λ_1 and scalars α and λ_2 which parameterize the price of risk. In total, there are 28 parameters and their estimated posterior modes are reported in Table 1.

Table 1

Posterior Parameter Estimates 1971:Q2-2011:Q1			
Factor processes			
$F^Q_{1,1}$	1.005	C_{21}	-0.018
$F^Q_{2,2}$	0.880	C_{31}	-0.001
$F^Q_{3,3}$	0.876	C_{32}	-0.032
C_1	0.014	$Q_{1,1}$	0.012
C_2	0.040	$Q_{2,2}$	0.039
C_3	0.012	$Q_{3,3}$	0.013
Maturity specific disturbances		Short rate constant r_t	
σ_v	0.012	δ_0	0.38
Risk Premia Parameters			
μ^P_1	-0.33	$\Lambda_{X,21}$	-363.2
μ^P_2	-0.004	$\Lambda_{X,22}$	-157.8
μ^P_3	0.002	$\Lambda_{X,23}$	-7.85
$\Lambda_{X,11}$	-423.5	$\Lambda_{X,31}$	-592.8
$\Lambda_{X,12}$	1488.2	$\Lambda_{X,32}$	-289.4
$\Lambda_{X,13}$	1953.7	$\Lambda_{X,33}$	-270.8
λ_2	-177.6	α	0.89
Log likelihood at $\widehat{\theta}$: 9985			

Since the factors are latent and have no particular interpretation, most of the parameters have little meaning when viewed in isolation. It is interesting to note though, that the second and third eigenvalues of the factor process under the risk neutral dynamic are very similar, i.e. *0.880* and *0.876*. This result is an implication of the restrictions implied by the model on the joint distribution of the cross-sectional dispersion of expectations and bond price dynamics mentioned above. If the three factors had very different persistence under the risk neutral dynamics it would be easy for traders to filter them out individually from bond prices since each factor would then have a very different impact on the shape of the yield curve. But if it would be possible to get very precise estimates of the latent factors from the yield curve, traders would filter out the factors perfectly and the cross-sectional dispersion of expectations would be too concentrated relative to the survey data. That two eigenvalues are close under the risk neutral dynamics makes the impact of the two corresponding factors difficult to disentangle. However, since the two factors have different persistence under the physical dynamics, it is still important for traders to try to form separate estimates of the two factors since they have different implications for future bond yields.

The implied dispersion of expectations at the posterior mode is somewhat lower than what is observed in the data. The one-quarter-ahead forecasts of the short interest rate has a cross-sectional standard deviation of about 26 basis points, as compared to the 40 basis points sample average in the data. That the fit of the cross-sectional variance of expectations is not perfect suggest that there is a trade-off between fitting the cross-sectional dispersion and the model's ability to fit other aspects of the data such as bond price dynamics. This provides more evidence that the joint restrictions on the cross-sectional dispersion of expectations and bond price dynamics that are present in a rational framework where traders use the information in bond prices efficiently has some empirical bite.

A. Historical decompositions

We can use the estimated model to measure how large the speculative terms have been historically. Speculative dynamics are driven by individual traders attempting to exploit what they perceive to be other traders expectation errors. As explained above, such errors must be orthogonal to public information available to all traders in real time, such as bond prices. However, as econometricians, we have at least one advantage compared to the traders in the model, and that is that we have access to a full sample of data and can use information from period $t+1, t+2, ...$ and so on, to form an estimate of the speculative term in period t. The Kalman smoother (see for instance Durbin and Koopman 2002) can be used to back out an estimate of the state X_t conditional on

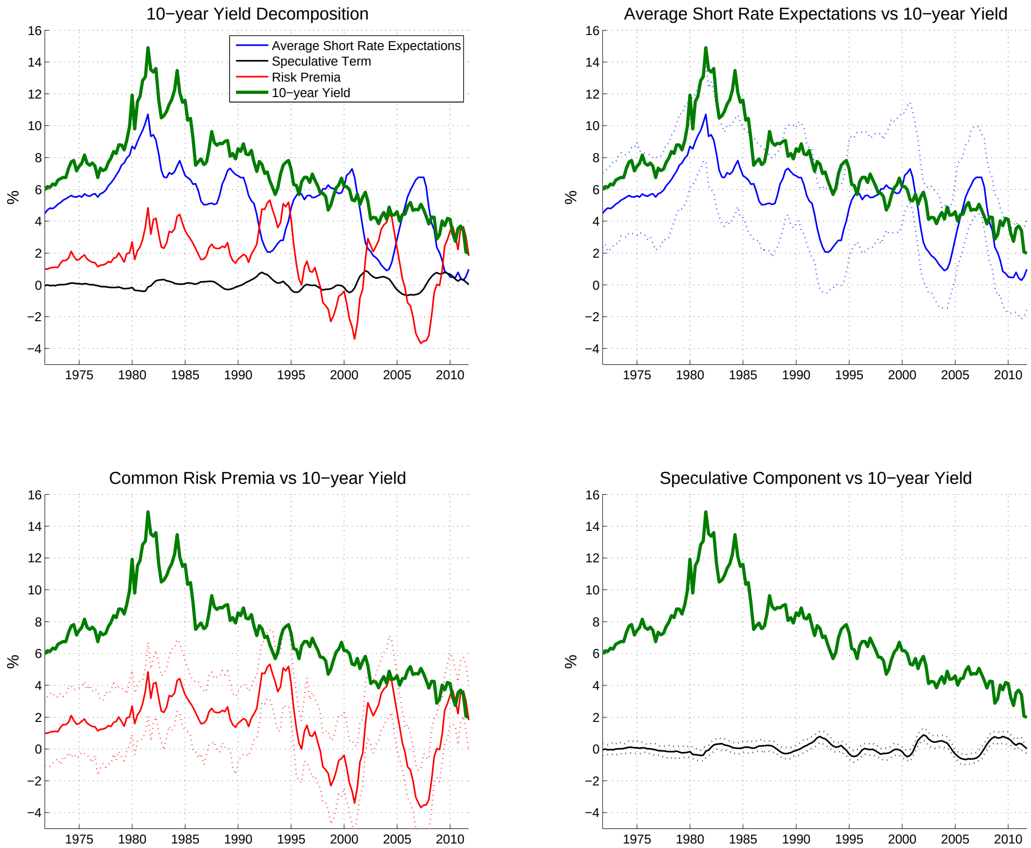


Figure 1: Individual terms in decomposition of the 10 year yield with 95 percent probability intervals.

the entire history of observables. Since the speculative components of the term structure are linear functions of the state, the backed out history $E\left[X^T \mid Z^T\right]$ can be used to construct estimates of the historical contribution to the bond yields by the individual components of the decompositions (36) and (39).

In the top left panel of Figure 1, we have plotted the history of the 10 year yield together with a decomposition, splitting the yield into the terms based on average expectations about future short rates, common risk premia and a speculative term. The individual terms are plotted with 95% probability intervals against the 10-year yield in the remaining three panels. It is clear form the figure that most of the variation in yields are driven by variation in average first order short rate expectations. The common risk premia term is the second most important term, but the speculative term is also quantitatively important. At the posterior median, the speculative term at times accounts for up to a full percentage point of yields.

Speculative dynamics are present at all maturities $n > 2$, but are quantitatively more important in medium- to long-maturity bonds. This is illustrated in Figure 2 where the estimated speculative component in the 1- 5- and 10-year yields are plotted. The speculative component is almost perfectly positively correlated across maturities and appears to be well-represented by a one-factor structure. The speculative term is most volatile in the 5-year bond yield, but only marginally more so than for the 10-year bond. In comparison, the volatility of the speculative term in the 1-year bond is substantially less volatile and never accounts for more than half a percentage point of the 1 year yield in the sample. This pattern across maturities is a function of that the number of orders of expectations that matter for a bond's price increases with maturity and the absolute magnitude of the speculative term is increasing monotonically in bond prices. However, to compute bond yields, we divide bond prices with the maturity of the bond and for bond yields with maturities beyond about 5 years this effect dominates the accumulation of higher order expectations.

The speculative component is positive when higher order expectations about future bond yields are above first order expectations and negative when they are below. If we look at the time series for the speculative components in Figure 2 we can see that at all maturities it reaches its minimum half-way through 2006 and stays low until the beginning of the financial crisis. The model thus suggests that before the crisis hit, traders thought that future bond yields would be lower than their expectation about the average prediction about future bond yields. One interpretation consistent with this evidence is that traders at the time expected both the crisis and the low interest rates that followed, but still did not believe that other traders shared this assessment.

Since future bond yields have both an expected short rate component as well as an expected risk premia component, we can decompose the speculative component further into speculation about future short rates and speculation about future risk premia, i.e. compute the third and fourth term in (39) separately. The result is shown in Figure 3 where the total speculative component (black line) is plotted together with the term due to speculation about future short rates (blue line) and the 95% probability interval. Speculation about future risk premia is thus the difference between

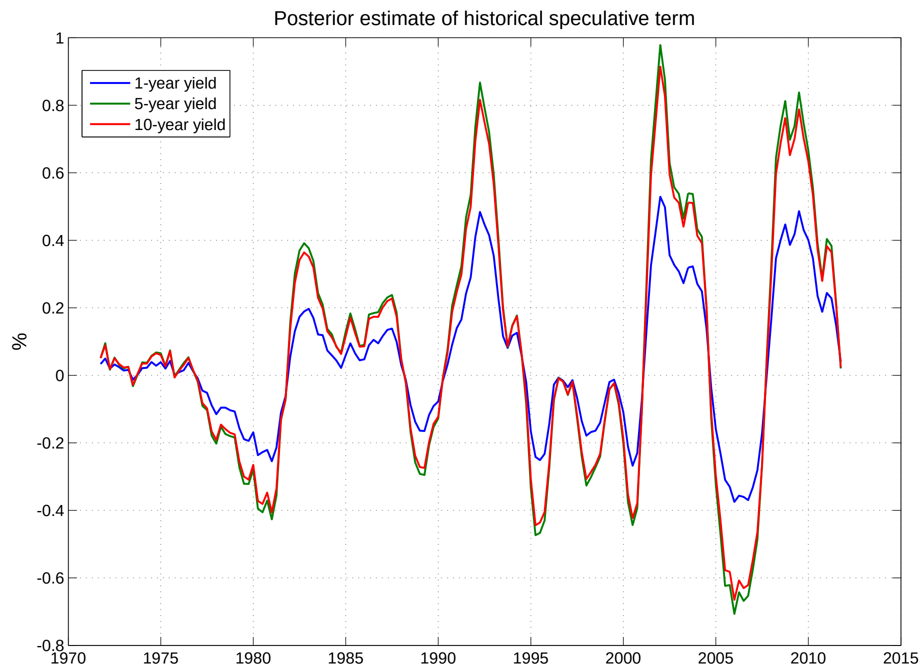


Figure 2: Individual terms in decomposition of the 10 year yield with 95 percent prob intervals.

the black and the blue line.

B. What drives speculative dynamics?

In order to address the question of what drives speculative dynamics we can decompose the variance of the speculative terms into four orthogonal sources: The three innovations to the exogenous factors in x_t and the maturity specific disturbances $\mathbf{v}_t$. The results of this exercise is displayed in Table 2 where the variance decompositions of the 1-, 5- and 10 years yields as well as the first three principal components are presented. The variance decomposition reveals that while the shocks to the third factor explains most of the variance of yields of all maturities, the first shock explains the majority of the variance of the speculative component. It is also the first shock that explains most of the variance of the second principal component, i.e. the so called "slope" factor.

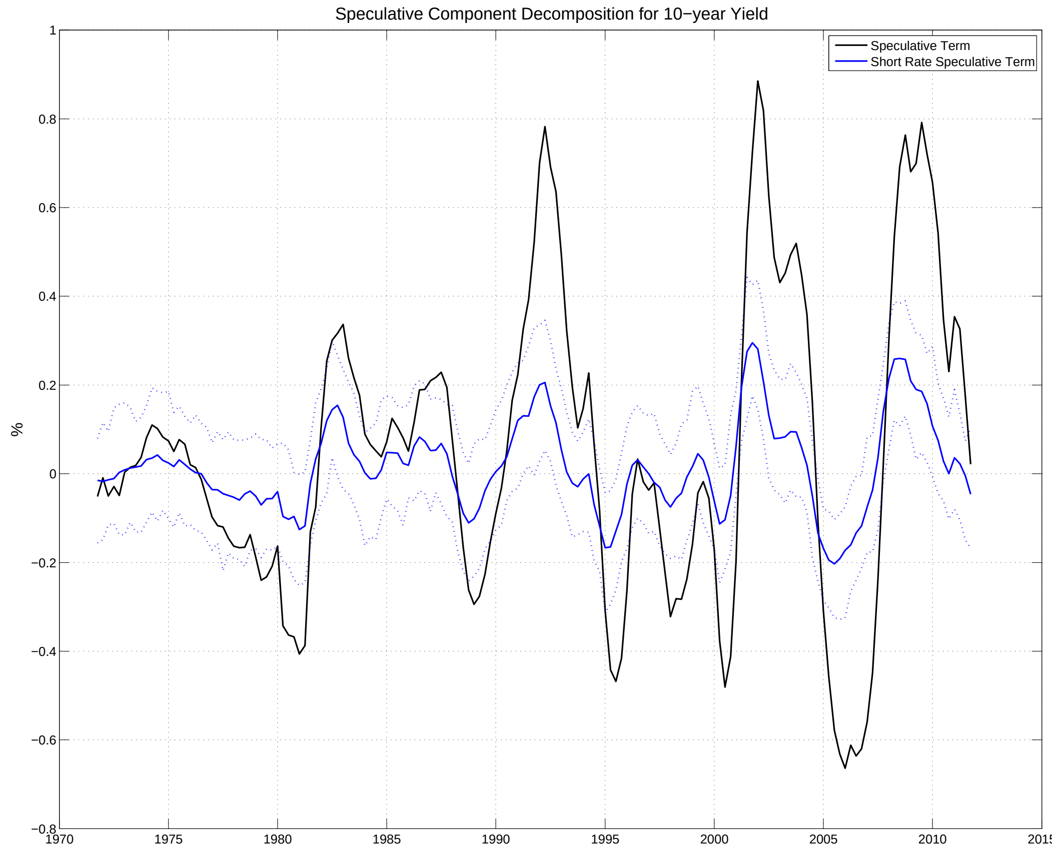


Figure 3: Decomposition of the 10 year yield speculative component.

Table 2 Variance decomposition

	y_t^4	y_t^{20}	y_t^{40}	pc_t^1	pc_t^2	pc_t^3	$spec_t^4$	$spec_t^{20}$	$spec_t^{40}$
ε_t^1	0.13	0.10	0.08	0.13	0.50	0.15	0.61	0.61	0.62
ε_t^2	0.09	0.10	0.11	0.11	0.05	0.00	0.01	0.01	0.02
ε_t^3	0.65	0.65	0.62	0.75	0.44	0.80	0.26	0.26	0.25
$\mathbf{v}_t$	0.12	0.14	0.17	0.01	0.01	0.05	0.12	0.12	0.12

It would be interesting to rotate the model into an equivalent representation where the states are the principal components and analyze how each of the principal components affect the speculative term at different maturities. However, by construction, no such equivalent representation exists. Since the principal components are observable directly from the cross-section of yields, such a representation could not capture the speculative dynamics as these are orthogonal to current bond prices.

The maturity specific disturbances $\mathbf{v}_t$ explains a substantial fraction of the variance of bond yields of all maturities. It is worth bearing in mind though that these are not traditional "pricing errors". An innovation to v_t^n does not affect only y_t^n but also traders' estimate of the state since y_t^n is part of all traders'observation vector z_t^j. Since the innovation affects traders' estimate of the state, it will also indirectly affect bond yields of maturities other than n. Due to persistence in traders' estimates of the state, this effect will not be confined to period t but the shock v_t^n will also affect bond yields of all maturities for several periods. Our specification is thus not subject to the critique in Hamilton and Wu (2011) who argue that the independent white noise assumption of classical pricing errors is testable and rejected by the data in standard affine term structure models.

C. Comparison to full information model

Gaussian affine term structure models have been used by for instance Cochrane and Piazzesi (2005) and Joslin, Priebsch and Singleton (2011) to decompose the the term structure into risk premia and expected future short rates. It is interesting to analyze how allowing for speculative dynamics changes the historical estimates of risk premia. In Figure 4 we have plotted the posterior estimate of the risk premia in the 10-year bond extracted using our model with heterogeneously informed traders together with the risk premia extracted using the full information model of Joslin et al (2011) which our model nests as a special case. One may have suspected that allowing for speculative dynamics would reduce the role played by risk premia in explaining the variance of bond yields, i.e. that the speculative term would partially "crowd out" time varying risk premia. Interestingly, the opposite turns out to be the case and Figure 4 shows that the risk premia term in the speculative model is estimated to be more volatile than the risk premia component estimated using the standard full information affine three factor model. Abstracting from speculative dynamics may thus lead a researcher to underestimate the importance of risk premia in explaining variation in bond yields.

D. How useful are the trader-specific signals?

We can also use the estimated model to quantify how useful the trader-specific signals are in terms of helping traders to forecast future bond prices. Figure 5 plots the reduction in the conditional yield variance due to observing the trader-specific signals, as compared to conditioning only

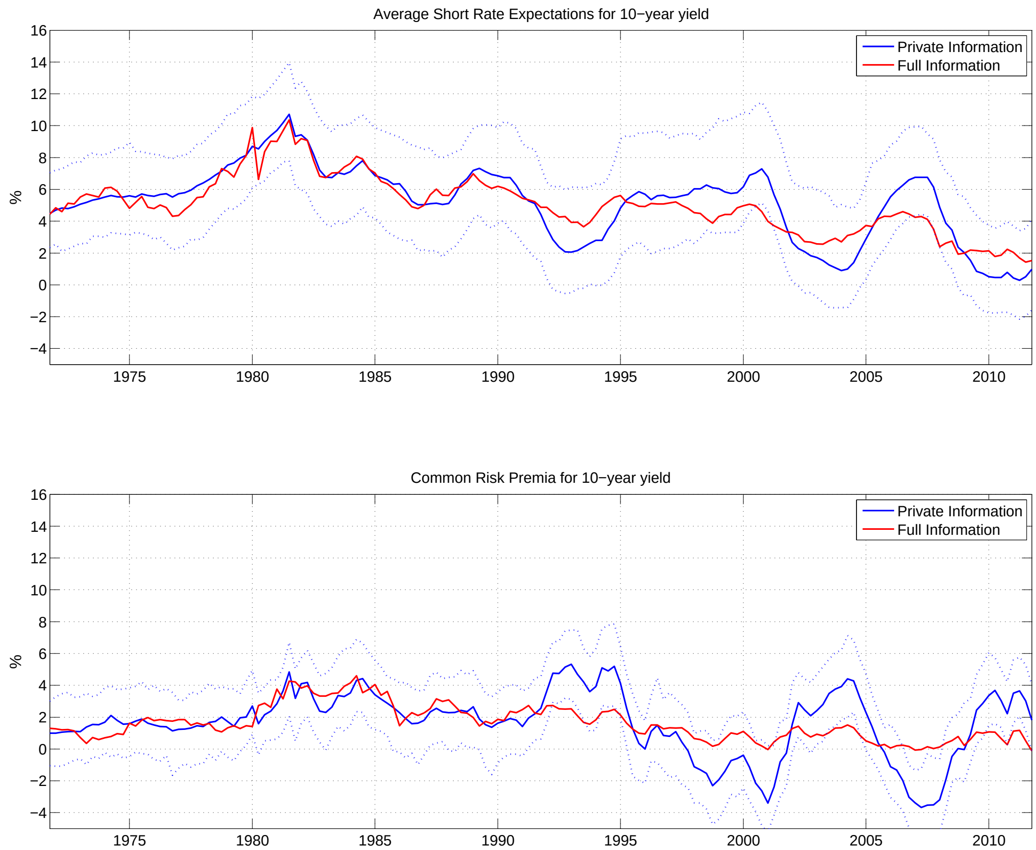


Figure 4: Comparison to Full-information model

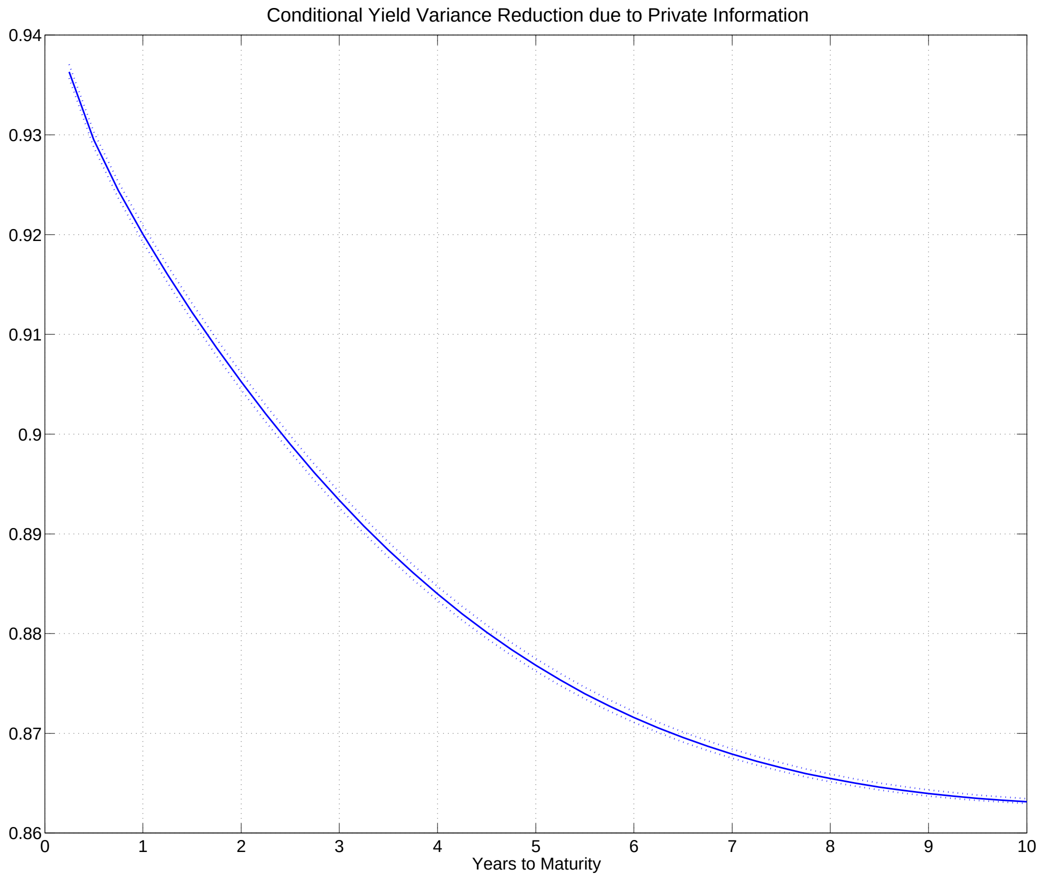
on publicly available bond yields. That is, Figure 5 plots the posterior estimate of the expression

$$\frac{E\left(y_{t+1}^{n}-E\left[y_{t+1}^{n}\mid\Omega_{t}^{j}\right]\right)^{2}}{E\left(y_{t+1}^{n}-E\left[y_{t+1}^{n}\mid\mathbf{y}^{t}\right]\right)^{2}} \tag{55}$$

for $n=1,2,...,40$ and where $y_t^n \equiv n^{-1}p_t^n$ and $\mathbf{y}^t$ is the history of bond yields up to period t. For short maturities, the conditional variance is reduced by about 6 per cent and for longer maturities the reduction reaches about 14 per cent for one quarter ahead forecasts of the 10-year yield.

V. Conclusions

In this paper we have presented an affine Gaussian framework for analyzing the term structure



of interest rates when rational traders have access to heterogeneous information. We showed that heterogeneous information introduces a speculative term in bond yields due to individual traders taking advantage of what they perceive to be inaccurate average expectations about future bond yields. Interpreting the history of US bond yields through the lens of the estimated model we found that speculative dynamics can be quantitatively important for medium to long maturity bonds. In the period of low interest rates in the first decade of the 2000s, the speculative component reaches a full percentage point at a time when the total yield on a ten year bond was below 5 per cent. We also find that allowing for speculative dynamics changes the estimates of historical risk premia. Interestingly, the posterior estimates of risk premia in the model with speculative dynamics are more volatile than in a nested affine model with perfectly informed traders. This suggests that it may be important to control for heterogeneous information and speculative dynamics to accurately extract measures of risk premia from bond yields. Since allowing for speculative dynamics affects the estimates of historical risk premia, allowing for speculative dynamics also have to the poential to change our view on what the economic forces are that drive time variation in risk premia. Similarly, the results presented here suggest that it may be important to control for speculative dynamics when extracting information about interest rates expectations from market bond prices.

In the model estimated here, traders form rational, i.e. model consistent, expectations. In the introduction, we argued mostly on *a priori* grounds that a rational framework has some advantages relative to alternative ways to introduce expectations heterogeneity based on bounded rationality. It is perhaps worth emphasizing a difference between the two approaches that has direct empirical implications. Since the traders in the model presented here use the information in equilibrium prices efficiently, the speculative term estimated here is orthogonal to bond prices in real time. This makes rational speculative behavior econometrically distinct from the speculative term in for instance Xiong and Yan (2010). They show that in their boundedly rational framework, speculative dynamics would appear to an econometrician as excess returns that are predictable conditioning only on bond prices, i.e. what has traditionally been called risk premia. Due to this distinct statistical property, the speculative component estimated here is really a different species altogether, relative to the classical components of the term structure and not simply risk premia relabeled. That rational speculative behavior must be orthogonal to public information in real time is a restriction that could, at least in principle, be used to distinguish rational from boundedly rational speculation empirically.

The stochastic discount factor framework presented here can also be used to price other asset classes. In the present paper, we found that speculative dynamics were quantitatively important in bond yields even though the value at maturity of a zero-coupon default-free bond is known with certainty and the only source of uncertainty is future market discount rates. The prices of other classes of assets, such as stocks and corporate bonds, also depend on expectations about future discount rates, but are also subject to additional sources of uncertainty due to stochastic cash-flows and the probability of default. It seems plausible that speculative dynamics could be even more important in those asset classes where prices depend on a richer set of variables.

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A An Affine Framework for Term Structure Models with Private Information.

In this appendix we derive the recursions for bond prices, we show how we solve the model following the algorithm in Nimark (2012) and provide additional details on the computation of the likelihood.

A. Derivations under the Physical Measure

We first derive recursions for bond prices. For a one-period bond (n=1) it follows that:

$$P_t^1 = E_t[M_{t+1}^j|\Omega(j)] = \exp(-r_t) \tag{56}$$

$$= \exp(-\delta_0 - \delta_X' X_t) \tag{57}$$

Matching coefficients leads to $A_1 = -\delta_0$ and $B_1 = -\delta_X$.

Now we show the recursion for any maturity n. The log price of a bond with $n+1$ periods to maturity is given by:

$$\begin{aligned} p_t^{n+1} &= \log E_t\left[M_{t+1}^j P_{t+1}^n|\Omega_t^j\right] \\ &= -r_t - \frac{1}{2}\Lambda_t^{j\prime}\Sigma_a\Lambda_t^j + A_n + \log E_t\left[\exp(-\Lambda_t^{j\prime}\mathbf{a}_{t+1}^j + B_n' X_{t+1} + \sigma_{v,n}v_{t+1}^n)|\Omega_t^j\right] \\ &= -r_t - \frac{1}{2}\Lambda_t^{j\prime}\Sigma_a\Lambda_t^j + A_n + B_n'\mu^P \\ &\quad + \log E_t\left[\exp(-\Lambda_t^{j\prime}\mathbf{a}_{t+1}^j + B_n'\mathbf{M}X_t + B_n'\mathbf{N}\mathbf{u}_{t+1}) + \sigma_{v,n}v_{t+1}^n)|\Omega_t^j\right] \end{aligned} \tag{58}$$

To compute this log price we first concentrate on the term containing the expectation in (58). Let V_n be the vector that selects the relevant maturity specific pricing error from $\mathbf{u}_t$. This is defined such that $V_n\ \mathbf{u}_t \equiv \sigma_{n,t}v_t^n$.

Also, note that the conditional variance of $\mathbf{a}_{t+1}^j$ is given by

$$\Sigma_a = \Sigma_{t+1|t} + \tilde{Q}\tilde{Q}' \tag{59}$$

Where we have used the fact that

$$\mathbf{a}_{t+1}^j = \mathbf{M}\left[X_t - E(X_t|\Omega_t^j)\right] + \mathbf{N}\mathbf{u}_{t+1} + \tilde{Q}\eta_{t+1}^j \tag{60}$$

where we define $\tilde{Q} = \begin{bmatrix} Q \\ \mathbf{0} \end{bmatrix}$

Then adding and subtracting $B_n'\mathbf{M}E[X_t|\Omega_t^j]$ and substituting the expression for $\mathbf{a}_{t+1}^j$ in (60) we obtain:

$$\begin{aligned}
& \log E_t\left[\exp(-\Lambda_t^{j\prime}\mathbf{a}_{t+1}^j + B_n'\mathbf{M}X_t + B_n'\mathbf{N}\mathbf{u}_{t+1} + \sigma_{v,n}v_{t+1}^n)|\Omega_t^j\right] \\
= \; & B_n'\mathbf{M}E[X_t|\Omega_t^j] \\
& + \log E_t\left[\exp((B_n' - \Lambda_t^{j\prime})\mathbf{M}(X_t - E(X_t|\Omega_t^j) + (B_n' - \Lambda_t^{j\prime})\mathbf{N}\mathbf{u}_{t+1} - \Lambda_t^{j\prime}\tilde{Q}\eta_{t+1}^j + V_n\mathbf{u}_{t+1}|\Omega_t^j\right] \\
= \; & B_n'\mathbf{M}E[X_t|\Omega_t^j] + (B_n' - \Lambda_t^{j,\prime})\mathbf{N}V_n' \\
& + \frac{1}{2}\left[(B_n' - \Lambda_t^{j\prime})(\mathbf{M}\Sigma_{t|t}\mathbf{M} + \mathbf{N}\mathbf{N}')'(B_n' - \Lambda_t^{j\prime})' + V_nV_n' + \Lambda_t^{j\prime}\tilde{Q}\tilde{Q}'\Lambda_t^j\right] \\
= \; & B_n'\mathbf{M}E[X_t|\Omega_t^j] + (B_n' - \Lambda_t^{j,\prime})\mathbf{N}V_n' \\
& + \frac{1}{2}\left[(B_n' - \Lambda_t^{j\prime})\Sigma_{t+1|t}(B_n' - \Lambda_t^{j\prime})' + V_nV_n' + \Lambda_t^{j\prime}\tilde{Q}\tilde{Q}'\Lambda_t^j\right] \qquad (61)
\end{aligned}$$

Plugging the evaluated expectation (61) back into the log price expression, using the expression for the conditional variance of $\mathbf{a}_{t+1}^j$ and plugging in the affine market price of risk parametrization (14) we obtain that:

$$\begin{aligned}
p_t^{n+1} \; = \; & -\delta_0 + A_n + B_n'\mu^P \\
& + \frac{1}{2}\left[(B_n' - \Lambda_t^{j\prime})\Sigma_{t+1|t}(B_n' - \Lambda_t^{j\prime})' + V_nV_n' + \Lambda_t^{j\prime}\tilde{Q}\tilde{Q}'\Lambda_t^j - \Lambda_t^{j\prime}\Sigma_a\Lambda_t^j\right] \\
& + (B_n' - \Lambda_t^{j\prime})\mathbf{N}V_n' - \delta_X'X_t + B_n'\mathbf{M}E[X_t|\Omega_t^j] \\
= \; & -\delta_0 + A_n + B_n'\mu^P \\
& + \frac{1}{2}\left[B_n'\Sigma_{t+1|t}B_n' + V_nV_n'\right] - B_n'\Sigma_{t+1|t}\Lambda_t^j + B_n'\mathbf{N}V_n' - V_n\mathbf{N}'\Lambda_t^j \\
& - \delta_X'X_t + B_n'\mathbf{M}E[X_t|\Omega_t^j] \\
= \; & -\delta_0 + A_n + B_n'\mu^P + \frac{1}{2}\left[B_n'\Sigma_{t+1|t}B_n' + V_nV_n'\right] - B_n'\Sigma_{t+1|t}\Lambda_0 + B_n'\mathbf{N}V_n' - V_n\mathbf{N}'\Lambda_0 \qquad (62) \\
& - \delta_X'X_t + B_n'\mathbf{M}E[X_t|\Omega_t^j] - (B_n'\Sigma_{t+1|t} + V_n\mathbf{N}')\Lambda_X X_t^j
\end{aligned}$$

The no-arbitrage condition (9) has to hold for all traders at all times. This implies that we could choose any trader j's state X_t^j as being the state variable that bond prices are a function of. However, the most convenient choice from a modeling perspective is to let the "average" trader's SDF price bonds. That is, we will let bonds be priced by the SDF of the fictional trader who's state X_t^j is equal to (25). Also noting that $E[X_t|\Omega_t^j] = HX_t$ then it follows that A_n and B_n follow the recursions:

$$\begin{aligned} A_{n+1} &= -\delta_0 + A_n + B_n'\mu^P \\ &+\frac{1}{2}\left[B_n'\Sigma_{t+1|t}B_n' + V_nV_n'\right] - (B_n'\Sigma_{t+1|t} + V_n\mathbf{N}')\Lambda_0 + B_n'\mathbf{N}V_n' \end{aligned} \tag{63}$$

$$B_{n+1}' = B_n'\mathbf{M}H - \delta_X' - (B_n'\Sigma_{t+1|t} + V_n\mathbf{N}')\Lambda_X \tag{64}$$

B. Derivations under the risk-neutral measure Q

Next, we derive the recursion for bond prices under the risk neutral measure. Under such measure, the price of a bond with maturity $n+1$ is computed by discounted its expected next period price by the risk-free rate:

$$\begin{aligned} p_t^{n+1} &= \log E^Q\left[e^{-r_t}P_{t+1}^n|\Omega_t^j\right] \\ &= -r_t + A_n^Q + \log E^Q\left[\exp(B_n^{Q\prime}X_{t+1} + \sigma_{v,n}v_{t+1}^{Q,n})|\Omega_t^j\right] \\ &= -r_t + A_n^Q + B_n^{Q\prime}\mu^Q + \log E^Q\left[\exp(B_n^{Q\prime}\mathbf{M}^QX_t + B_n^{Q\prime}\mathbf{N}\mathbf{u}_{t+1}^Q) + \sigma_{v,n}v_{t+1}^{Q,n})|\Omega_t^j\right] \end{aligned} \tag{65}$$

To compute this price we first concentrate on the term containing the expectation.

$$\begin{aligned} & \log E_t\left[\exp(B_n'\mathbf{M}^QX_t + B_n'\mathbf{N}\mathbf{u}_{t+1}^Q) + \sigma_{v,n}v_{t+1}^{Q,n})|\Omega_t^j\right] \\ = \;& \log E_t\left[\exp\left[B_n'\mathbf{M}^Q(X_t - E(X_t|\Omega_t^j) + B_n'\mathbf{M}^QE[X_t|\Omega_t^j] + B_n'\mathbf{N}\mathbf{u}_{t+1}^Q + \sigma_{v,n}v_{t+1}^{Q,n}\right]|\Omega_t^j\right] \\ = \;& B_n'\mathbf{M}E[X_t|\Omega_t^j] + \log E_t[\exp\left[B_n'\mathbf{M}^Q(X_t - E(X_t|\Omega_t^j) + B_n'\mathbf{N}\mathbf{u}_{t+1}^Q + V_n\mathbf{u}_{t+1}^Q\right]|\Omega_t^j] \\ = \;& B_n'\mathbf{M}^QE[X_t|\Omega_t^j] + \frac{1}{2}\left[B_n'(\mathbf{M}^Q\Sigma_{t|t}\mathbf{M}^Q + \mathbf{N}\mathbf{N}')'B_n{}' + V_nV_n'\right] + B_n'\mathbf{N}V_n' \\ = \;& B_n'\mathbf{M}^QE[X_t|\Omega_t^j] + \frac{1}{2}\left[B_n'\Sigma_{t+1|t}^QB_n + V_nV_n'\right] + B_n'\mathbf{N}V_n' \end{aligned}$$

The log bond price is then given by:

$$p_t^{n+1} = -\delta_0 + A_n^Q + B_n^{Q\prime}\mu^Q - \delta_X'X_t + B_n^{Q\prime}\mathbf{M}^QE[X_t|\Omega_t^j] + \frac{1}{2}\left[B_n^{Q\prime}\Sigma_{t+1|t}^QB_n^Q + V_nV_n'\right] + B_n^{Q\prime}\mathbf{N}V_n' \tag{66}$$

so that applying analogous arguments as in the derivation of the expressions under the physical measure the coefficients A_n^Q and B_n^Q can be found by the recursion

$$A_{n+1}^Q = -\delta_0 + A_n^Q + B_n^{Q\prime}\mu^Q + \frac{1}{2}\left[B_n^{Q\prime}\Sigma_{t+1|t}^QB_n^Q + V_nV_n'\right] + B_n^{Q\prime}\mathbf{N}V_n' \tag{67}$$

$$B_{n+1}^{Q\prime} = B_n^{Q\prime}\mathbf{M}^QH - \delta_X' \tag{68}$$

starting from

$$A_1^Q = -\delta_0 \tag{69}$$

$$B_1^Q = \begin{bmatrix} -\delta_X' & \mathbf{0} \end{bmatrix} \tag{70}$$

since $p_t^1 = -r_t$.

C. Identification

The loadings derived under risk neutral and physical dynamics are by construction equal to each other. That is $A_n^Q = A_n$ and $B_n^Q = B_n$ for all n. Following Joslin, Singleton and Zhu (2011) we impose restrictions on the risk neutral model to make sure the model is identified under full information. Then matching this recursions we show how one can solve for the unconditional mean of the state.

C.1. Matching recursions for B

$$\begin{aligned} B_n'\mathbf{M}H - \delta_X' - (B_n'\Sigma_{t+1|t} + V_n\mathbf{N}')\Lambda_X &= B_n'\mathbf{M}^Q H - \delta_X' \\ B_n'\mathbf{M}H - (B_n'\Sigma_{t+1|t} + V_n\mathbf{N}')\Lambda_X &= B_n'\mathbf{M}^Q H \end{aligned} \tag{71}$$

here we impose the identifying restriction that that the upper left hand 3×3 sub matrix of M^Q is diagonal with decreasing elements. This ensures that the model is identified under full information.

C.2. Matching recursions for A

$$B_n'\mu^P + \frac{1}{2}\left[B_n'\Sigma_{t+1|t}B_n'\right] - (B_n'\Sigma_{t+1|t} + V_n\mathbf{N}')\Lambda_0 = B_n'\mu^Q + \frac{1}{2}\left[B_n'\Sigma_{t+1|t}^Q B_n\right] \tag{72}$$

When we impose the identifying restriction that $\mu^Q = 0$ which ensures that the model is identified under full information this allows us to solve for for $B_n'\mu^P$

$$\begin{aligned} & B_n'\mu^P + \frac{1}{2}\left[B_n'\mathbf{M}^P\Sigma_{t|t}\mathbf{M}^P B_n + B_n'\mathbf{N}\mathbf{N}'B_n\right] - (B_n'\Sigma_{t+1|t} + V_n\mathbf{N}')\Lambda_0 \\ = \; & \frac{1}{2}\left[B_n'\mathbf{M}^Q\Sigma_{t|t}\mathbf{M}^Q B_n + B_n'\mathbf{N}\mathbf{N}'B_n\right] \end{aligned}$$

which equivalently can be written as

$$\begin{aligned} & B_n'\mu^P + \frac{1}{2}\left[B_n'\mathbf{M}\Sigma_{t|t}\mathbf{M}B_n\right] - (B_n'\Sigma_{t+1|t} + V_n\mathbf{N}')\Lambda_0 \\ = & \frac{1}{2}\left[B_n'\mathbf{M}^Q\Sigma_{t|t}\mathbf{M}^Q B_n\right] \end{aligned} \tag{73}$$

Rearranging gives

$$B_n'\mu^P = \frac{1}{2}\left[B_n'\mathbf{M}^Q\Sigma_{t|t}\mathbf{M}^Q B_n\right] - \frac{1}{2}\left[B_n'\mathbf{M}\Sigma_{t|t}\mathbf{M}B_n\right] + (B_n'\Sigma_{t+1|t} + V_n\mathbf{N}')\Lambda_0 \tag{74}$$

B Solving the model

Solving the model implies finding a law of motion for the higher order expectations of x_t of the form

$$X_{t+1} = \mu^X + \mathbf{M}X_t + \mathbf{N}\mathbf{u}_{t+1} \tag{75}$$

where

$$X_t \equiv \begin{bmatrix} x_t^{(0)} \\ x_t^{(1)} \\ \vdots \\ x_t^{(\overline{k})} \end{bmatrix}, \mathbf{u}_t = \begin{bmatrix} \varepsilon_t \\ \mathbf{v}_t \end{bmatrix}$$

That is, to solve the model, we need to find the matrices $\mathbf{M}$ and $\mathbf{N}$ as functions of the parameters governing the short rate process, the maturity specific disturbances and the idiosyncratic noise shocks. The integer $\overline{k}$ is the maximum order of expectation considered and can be chosen to achieve an arbitrarily close approximation to the limit as $\overline{k} \to \infty$. Here, a brief overview of the method is given, but the reader is referred to Nimark (2011) for more details on the solution method.

First, common knowledge of the model can be used to pin down the law of motion for the vector X_t containing the hierarchy of higher order expectations of x_t. Rational, i.e. model consistent, expectations of x_t thus implies a law of motion for average expectations $x_t^{(1)}$ which can then be treated as a new stochastic process. Knowledge that other traders are rational, means that second order expectations $x_t^{(2)}$ are determined by the average across traders of the rational expectations of the stochastic process $x_t^{(1)}$. Third order expectations $x_t^{(3)}$ are then the average of the rational expectation of the process $x_t^{(2)}$, and so on. Imposing this structure on all orders of expectations allows us to find the matrices $\mathbf{M}$ and $\mathbf{N}$. Section A below describes how this is implemented in practice.

Second, the method exploits that the importance of higher order expectations are decreasing with the order of expectation. This has two components:

(i) The variance of higher order expectations of the factors x_t are bounded by the variance of the true process, or more generally, the variance of $k+1$ order expectation cannot be larger than the variance of a k order expectation

$$cov\left(x_t^{(k+1)}\right) \leq cov\left(x_t^{(k)}\right) \tag{76}$$

To see why, note that by the identity

$$x_t^{(k)} \equiv x_t^{(k+1)} + \mathbf{e}_t^{(k+1)} \tag{77}$$

and the fact that since $x_t^{(k+1)}$ is the average of an optimal estimate of $x_t^{(k)}$ the $k=1$ order error $\mathbf{e}_t^{(k+1)}$ must be orthogonal to $x_t^{(k+1)}$ we have that

$$cov\left(x_t^{(k)}\right) = cov\left(x_t^{(k+1)}\right) + cov\left(\mathbf{e}_t^{(k+1)}\right). \tag{78}$$

Since

$$cov\left(\mathbf{e}_t^{(k+1)}\right) \geq \mathbf{0} \tag{79}$$

the inequality (76) follows immediately. (This is an abbreviated version of a more formal proof available in Nimark (2011).)

That the variances of higher order expectations of the factors are bounded is not sufficient for an accurate finite dimensional solution. We also need (ii) that the impact of the expectations of the factors on bond yields are decreasing "fast enough" with the order of expectation. The proof of this result is somewhat involved and readers are referred to the original reference for a proof.

A. The law of motion of higher order expectations of the factors

To find the law of motion for the hierarchy of expectations X_t we use the following strategy. For given $\mathbf{M}, \mathbf{N}$ in (75) and B_n' in (27) we will derive the law of motion for trader j's expectations of X_t, denoted $X_{t|t}^j \equiv E\left[X_t \mid \Omega_t^j\right]$. First, write the vector of signals z_t^j as a function of the state, the aggregate shocks and the idiosyncratic shocks

$$z_t^j \equiv \begin{bmatrix} x_t^j \\ r_t \\ \mathbf{y}_t \end{bmatrix} \tag{80}$$

$$= \mu_z + DX_t + R \begin{bmatrix} \mathbf{u}_t \\ \eta_t^j \end{bmatrix} \tag{81}$$

where the matrix D is given by

$$D = \begin{bmatrix} I_3 \quad \mathbf{0} \\ B_1' \\ \vdots \\ N^{-1}B_N' \end{bmatrix} \tag{82}$$

and R can be partitioned conformably to the aggregate and idiosyncratic shocks

$$R = \begin{bmatrix} R_u & R_\eta \end{bmatrix}. \tag{83}$$

The matrix R_u picks out the appropriate maturity specific shocks v_t^n from the vector of aggregate shocks $\mathbf{u}_t$ and R_η adds the idiosyncratic shocks $Q\eta_t^j$ to the exogenous state x_t to form the trader j specific signal vector x_t^j, i.e.

$$R_\eta = \begin{bmatrix} Q \\ \mathbf{0} \end{bmatrix}$$

Trader j's updating equation of the state $X_{t|t}^j$ estimate will then follow

$$X_{t|t}^j = \mu^X + \mathbf{M}X_{t-1|t-1}^j + K\left(z_t^j - D(\mu^X + \mathbf{M}X_{t-1|t-1}^j)\right) \tag{84}$$

Rewriting the observables vector z_t^j as a function of the lagged state and current period innovations and taking averages across traders using that $\int \zeta_t(j)dj = 0$ yields

$$X_{t|t} = \mu^X + \mathbf{M}X_{t-1|t-1} + K\left(D\left(\mu^X + \mathbf{M}X_{t-1}\right) + (D\mathbf{N} + R_u)\,\mathbf{u}_t - D(\mu^X + \mathbf{M}X_{t-1|t-1}^j)\right) \tag{85}$$

$$= \mu^X + (\mathbf{M} - KD\mathbf{M})\,X_{t-1|t-1} + KD\mathbf{M}X_{t-1} + K\,(D\mathbf{N} + R_u)\,\mathbf{u}_t \tag{86}$$

Appending the average updating equation to the exogenous state gives us the conjectured form of the law of motion of $x_t^{(0:\bar{k})}$

$$\begin{bmatrix} x_t \\ X_{t|t} \end{bmatrix} = \mu^X + \mathbf{M}\begin{bmatrix} x_{t-1} \\ X_{t-1|t-1} \end{bmatrix} + \mathbf{N}\mathbf{u}_t \tag{87}$$

where $\mathbf{M}$ and $\mathbf{N}$ are given by

$$\mathbf{M} = \begin{bmatrix} F^P & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{0}_{3\times 3} & \mathbf{0} \\ \mathbf{0} & [\mathbf{M} - KD\mathbf{M}]_- \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ [KD\mathbf{M}]_- \end{bmatrix} \tag{88}$$

$$\mathbf{N} = \begin{bmatrix} C & 0 \\ \mathbf{0} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ [K\,(D\mathbf{N} + R_u)]_- \end{bmatrix} \tag{89}$$

where $[\cdot]_{\_}$ indicates that the a last row or column has been canceled to make a the matrix $[\cdot]$ conformable, i.e. implementing that $x_t^{(k)} = 0 : k > \overline{k}$. The Kalman gain K in (84) is given by

$$K = \left(PD' + \mathbf{N}R_u\right)\left(DPD' + RR'\right)^{-1} \tag{90}$$

$$P = \mathbf{M}\left(P - \left(PD' + \mathbf{N}R_u\right)\left(DPD' + RR'\right)^{-1}\left(PD' + \mathbf{N}R_u\right)'\right)\mathbf{M}' + \mathbf{N}\mathbf{N}' \tag{91}$$

The model is solved by finding a fixed point that satisfies (27), (88), (89), (90) and (91).

C Computing the cross-sectional variance Σ_j

The idiosyncratic noise shocks η_t^j are white noise processes that are orthogonal across traders and to the aggregate shocks $\mathbf{v}_t$ and ε_t. This implies that the cross-sectional variance of expectations is equal to the part of the unconditional variance of trader j's expectations that is due to idiosyncratic shocks. This quantity can be computed by finding the variance of the estimates in trader j's updating equation (84), but with the aggregate shocks $\mathbf{v}_t$ and ε_t "switched off". The covariance Σ_j of trader j's state estimate due to idiosyncratic shocks is defined as

$$\Sigma_j \equiv E\left(X_{t|t}^j - X_{t|t}\right)\left(X_{t|t}^j - X_{t|t}\right)' \tag{92}$$

and given by the solution to the Lyaponov equation

$$\Sigma_j = (I - KD)\,\mathbf{M}\Sigma_j\mathbf{M}'\,(I - KD)' + KR_\eta R_\eta K'. \tag{93}$$

which can be found by simply iterating on (93).